POLLICOTT-RUELLE EQUALS SCATTERING

HAMID AL-SAQBAN AND TOBIAS WEICH

Abstract. We establish that on Cartan–Hadamard manifolds with strictly convex boundary and pinched negative curvature, the set of Pollicott–Ruelle resonances associated with the geodesic flow coincides with the set of scattering resonances defined via the scattering operator. Our proof relies on a generalization of a result of the second-named author to the Cartan-Hadamard setting: we show that all generalized resonant and co-resonant states of the flow have full topological support on the incoming and outgoing tails, respectively. We further use a microlocal transversality result, due to Y. Chaubet in dimension two and to M. Cekic, C. Guillarmou, and T. Lefeuvre in general, which ensures that the pullback of the meromorphically continued resolvent kernel to the boundary is well-posed, and yields the scattering operator.

1. Introduction

[1] The study of chaotic dynamics on negatively curved manifolds has long revealed deep connections between geometry, analysis, and dynamical systems. In the compact setting, geodesic flows on closed negatively curved manifolds are canonical examples of Anosov flows, characterized by their uniform hyperbolicity and rich spectral structure. Central to their analysis is the notion of *Pollicott–Ruelle resonances*, complex frequencies that encode the decay rates of dynamical correlations and govern the fine statistical properties of the flow. These resonances form a discrete set in the complex plane and have been extensively studied via microlocal methods.

In recent years, attention has turned toward understanding such dynamical spectra in the non-compact setting, particularly for open systems where geodesics may escape to infinity. In these contexts, one may still define Pollicott–Ruelle resonances, and this was achieved in seminal work by Dyatlov and Guillarmou, who constructed a robust microlocal theory for resonances in open hyperbolic systems. Concurrently, these developments have been paralleled by advances in inverse problems and scattering theory.

Scattering operators also admit meromorphic extensions in these settings. Intriguingly, their poles–scattering resonances–bear a close relationship to the dynamical resonances mentioned above. In recent work by Chaubet and by Cekic, Guillarmou, and Lefeuvre, it has been shown that for Anosov-type compact manifolds with strictly convex boundaries and hyperbolic trapped sets, the Schwartz kernel of the scattering operator can be obtained by the pullback of the dynamical resolvent kernel to the boundary. This identification allows for a precise comparison between the two notions of resonance.

The current paper contributes to this confluence of ideas by establishing that Pollicott–Ruelle resonances and scattering resonances coincide. Specifically, we work on Cartan–Hadamard manifolds of pinched negative curvature, modulo a discrete group action, and consider a compact subregion of the unit tangent bundle with strictly

[1]Last updated on May 23, 2025.

convex boundary. Under the assumption that the geodesic flow is hyperbolic on its trapped set, we prove that the poles of the meromorphic continuation of the boundary scattering operator match exactly with the Pollicott–Ruelle resonances of the flow.

Our approach is twofold: first, we generalize a result of the second-named author to show that generalized resonant and co-resonant states have full support on the incoming and outgoing tails, respectively. Second, using the aforementioend results on the pullback of the dynamical resolvent kernel to the boundary, we relate the scattering operator to the dynamical resolvent.

2. Notation and Main Results

Let $(\widetilde{M}, \tilde{g})$ be a Cartan–Hadamard manifold: a complete, simply connected Riemannian manifold of dimension $n \geq 2$ with sectional curvature uniformly pinched between two negative constants,

$$-b^2 \leq \mathrm{Sec}_{\tilde{g}} \leq -a^2 < 0,$$

and with bounded derivatives of the sectional curvature. Let $\Gamma < \mathrm{Isom}(\widetilde{M})$ be a torsion-free discrete group acting freely and properly discontinuously, and define the complete, non-compact manifold

$$M := \Gamma \backslash \widetilde{M}.$$

Let $\overline{\mathcal{U}} \subset SM$ be a compact manifold with boundary, with interior $\mathcal{U}$ and boundary $\partial\mathcal{U}$. Let X denote the geodesic vector field generating the flow $\varphi^t := e^{tX}$ on SM. We assume that the boundary $\partial\mathcal{U}$ is *strictly convex* in the sense that

$$x \in \partial\mathcal{U}, \quad X\rho(x) = 0 \quad \Longrightarrow \quad X^2\rho(x) < 0,$$

for some boundary defining function $\rho \in C^\infty(\overline{\mathcal{U}})$ satisfying $\rho > 0$ in $\mathcal{U}$, $\rho = 0$ on $\partial\mathcal{U}$, and $d\rho \neq 0$ on $\partial\mathcal{U}$. As stated in **[DG16]**, this definition is independent of the choice of boundary defining function ρ.

We define the *incoming and outgoing tails* by

$$\Gamma_\pm := \bigcap_{\pm t \geq 0} \varphi_t(\mathcal{U}), \qquad K := \Gamma_+ \cap \Gamma_- \subset \mathcal{U},$$

so that K consists of unit tangent vectors whose geodesics remain in $\mathcal{U}$ for all time. We assume that the trapped set K is:

(1) *hyperbolic*, i.e., for all $y \in K$, the tangent space admits a continuous, flow-invariant splitting

$$T_ySM = \mathbb{R}X(y) \oplus E_-(y) \oplus E_+(y),$$

where there exist constants $\nu, C > 0$ such that

$$\|d\varphi_t(y)v\| \leq Ce^{-\nu|t|}\|v\|, \quad \text{for all } \pm t \geq 0,\ v \in E_\mp(y),$$

(2) *compact*,

(3) and a *basic set* for the flow, meaning it is locally maximal, and the flow is topologically transitive on K.

Here, $\|\cdot\|$ denotes the Sasaki norm induced by g. Following Dyatlov–Guillarmou **[DG16]**, we perform all our analysis on a compact submanifold $\overline{\mathcal{U}} \subset SM$ containing K, with strictly convex boundary. In particular, their work shows that one can construct anisotropic Sobolev spaces adapted to the stable and unstable foliations, on which the operator

$$R(\lambda) := (X + \lambda)^{-1}$$

admits a meromorphic extension from $\mathrm{Re}(\lambda) \gg 1$ to $\mathbb{C}$ as a family of operators

$$R(\lambda) : C_c^\infty(\mathcal{U}) \to \mathcal{D}'(\mathcal{U}),$$

and the poles of this extension are called *Pollicott–Ruelle resonances*.

In this paper, we are concerned with two questions:

(1) What is the topological support of generalized resonant and co-resonant states associated to a given resonance $\lambda_0 \in \mathbb{C}$?
(2) Does the set of Pollicott–Ruelle resonances coincide with the set of poles of the so-called *scattering operator*?

Our first result addresses the first question, extending a result of the second-named author for compact manifolds admitting an Anosov flow **[Wei17]** to the setting described above:

Theorem 2.1 (Full Support of Generalized Resonant and Co-Resonant States)**.** *Let $\lambda_0 \in \mathbb{C}$ be a Pollicott–Ruelle resonance of the geodesic vector field X on $\mathcal{U}$. Then:*

- *If $u \in \mathcal{D}'(\mathcal{U})$ satisfies $(X+\lambda_0)^j u = 0$ for some $j \geq 1$, and $\mathrm{WF}(u) \subset E_+^*$, then*
$$\mathrm{supp}(u) = \Gamma_+.$$
- *If $v \in \mathcal{D}'(\mathcal{U})$ satisfies $(X^* + \overline{\lambda_0})^j v = 0$ for some $j \geq 1$, and $\mathrm{WF}(v) \subset E_-^*$, then*
$$\mathrm{supp}(v) = \Gamma_-.$$

To prove the first part of Theorem 2.1, we assume that a resonant state vanishes on a subset of Γ_+, and construct test distributions supported near the trapped set K, exploiting its local product structure and the smooth disintegration of the Liouville measure along strong stable leaves. Microlocal transversality of wavefront sets guarantees that the pairing with the resonant state vanishes. Then, the wavefront condition ensures that vanishing propagates forward along the flow. Since $\Gamma_+ = W^s(K)$, and the backward orbit of any point in Γ_+ accumulates on K, this forces global vanishing, contradicting nontriviality of the resonant state. The argument for the second part is analogous, with time reversed and stable/unstable directions exchanged.

In order to state our next result, we need additional preliminaries. Let $\partial_\pm \mathcal{U} \subset \partial\mathcal{U}$ denote the incoming and outgoing boundary components of $\mathcal{U}$, defined as

$$\partial_\pm \mathcal{U} := \{(x,v) \in \partial\mathcal{U} \mid \pm X\rho(x,v) < 0\},$$

where, as above, ρ is a boundary defining function for $\mathcal{U}$. These sets correspond to unit tangent vectors whose geodesics enter $(-)$ or exit $(+)$ the region $\mathcal{U}$. The boundary scattering map S_g associates to each $(x,v) \in \partial_-\mathcal{U}$ the point where the geodesic first exits $\mathcal{U}$:

$$S_g(x,v) := \varphi^g_{\tau_g(x,v)}(x,v), \quad \text{where } \tau_g(x,v) \text{ is the first exit time.}$$

This defines a *scattering map*, which is a diffeomorphism

$$S_g : \partial_-\mathcal{U} \setminus \Gamma_- \to \partial_+\mathcal{U} \setminus \Gamma_+.$$

The associated *scattering operator* is defined by pullback under S_g:

$$\mathcal{S}_g : C_c^\infty(\partial_+\mathcal{U} \setminus \Gamma_+) \to C_c^\infty(\partial_-\mathcal{U} \setminus \Gamma_-),$$
$$(\mathcal{S}_g f)(z) := f(S_g(z)).$$

Remark 2.2. The scattering map S_g determines, and is in turn determined by, the corresponding scattering operator $\mathcal{S}_g$; that is, the operator encodes the action of the map on boundary data, and vice versa, and we refer to **[CGL24**, Section 2.2**]** for a more thorough discussion on this topic.

Furthermore, the Schwartz kernel of the scattering operator $\mathcal{S}_g(\lambda)$ can be obtained via the Schwartz kernel of the meromorphic extension of the resolvent $R(\lambda)$, as follows. Let $R(\lambda; z, z')$ be the Schwartz kernel of $R(\lambda)$, a distribution on $\mathcal{U} \times \mathcal{U}$. Define the Schwartz kernel of the boundary scattering operator as

$$\mathcal{S}_g(\lambda)(y_-, y_+) := -(\iota_{\partial^-} \times \iota_{\partial^+})^* R(\lambda; y_-, y_+),$$

where $\iota_{\partial^\pm} : \partial_\pm \mathcal{U} \hookrightarrow \mathcal{U}$ are the inclusion maps. By results of Chaubet in dimension 2 [**Cha24**, Proposition 3.2] and Cekic–Guillarmou–Lefeuvre in general [**CGL24**, Lemma 2.7], this pullback is well-defined, and establishes the Schwartz kernel $\mathcal{S}_g(\lambda)$ of the scattering operator as a distribution on $\partial_- \mathcal{U} \times \partial_+ \mathcal{U}$.

We are now ready to state our second result:

Theorem 2.3. *Let $(M, g) = \Gamma \backslash \widetilde{M}$ be as above. Then the set of Pollicott–Ruelle resonances of the geodesic vector field X on $\mathcal{U}$ coincides with the set of poles of the meromorphic extension of the boundary scattering operator $\mathcal{S}_g(\lambda)$:*

$$\{\textit{Pollicott–Ruelle resonances of } X\} = \{\textit{poles of } \mathcal{S}_g(\lambda)\}.$$

This equivalence is proved by first showing, via Theorem 2.1, that the residues of the resolvent have full support on $\Gamma_- \times \Gamma_+$, and therefore restrict nontrivially to $\partial_- M \times \partial_+ M$, so every Pollicott–Ruelle resonance induces a pole of the scattering operator. The converse inclusion follows from the fact that the scattering kernel is defined as a pullback of the resolvent.

3. Proof of Theorem 2.1

Proof. Let $\lambda_0 \in \mathbb{C}$ be a Pollicott–Ruelle resonance of the operator X acting on distributions supported in $\mathcal{U}$, as defined via meromorphic continuation of the resolvent. Let $u \in \mathcal{D}'(\mathcal{U})$ be a generalized resonant state associated to λ_0. That is,

$$(X + \lambda_0)^j u = 0$$

for some integer $j \geq 1$, and

$$\mathrm{WF}(u) \subset E_+^*, \qquad \mathrm{supp}(u) \subset \Gamma_+.$$

The goal is to show that:

$$\mathrm{supp}(u) = \Gamma_+.$$

We now explain the strategy, which follows closely in outline the argument in [**Wei17**]. There, the proof proceeds by:

(1) Assuming the resonant state u vanishes on an open subset of SM,
(2) Constructing a distribution $\rho \in \mathcal{D}'(U)$, supported where $u = 0$, such that $\mathrm{WF}(\rho) \subset E_-^*$, and using the fact that $\mathrm{WF}(u) \subset E_+^*$ with $E_+^* \cap E_-^* = \{0\}$, to conclude $\langle u, \rho \rangle = 0$ by Hörmander's criterion for the multiplication of distributions,
(3) Using flow invariance of the wavefront set to propagate vanishing, along with topological transitivity, to derive a contradiction.

In the open hyperbolic setting, we adapt each of these steps using the local hyperbolic structure near the trapped set and by leveraging the leafwise smooth disintegration of the Liouville measure (available after lifting to the universal cover, thanks to [**PPS15**, Theorem 7.6]).

Assume for the sake of contradiction that $\mathrm{supp}(u) \subsetneq \Gamma_+$. Then there exists a point $x_0 \in \Gamma_+ \setminus \mathrm{supp}(u)$ and a relatively compact open neighborhood $W' \subset U$ of x_0 such that $\mathrm{supp}(u) \cap W' = \emptyset$. Define $W := W' \cap \Gamma_+$, so $u|_W = 0$.

Because $\Gamma_+ = W^u(K)$ (see [**Gui17**, Lemma 2.2] and its proof), and because K is hyperbolic, Γ_+ is locally laminated near K by weak unstable manifolds $W^{wu}(x)$. In a neighborhood $\mathcal{N}_K \subset \mathcal{U}$, one may construct local product coordinates $\mathcal{V} \subset \mathcal{N}_K$ of the form $\mathcal{V} \cong V^{ss} \times V^u$, where:

(1) $V^{ss} \subset W^{ss}(x)$ is a small neighborhood in the strong stable manifold of x,
(2) V^u is a transversal in the unstable direction.

Let $A \subset V^u$ be open. Following the notation set in [**Wei17**], we define *the stable tube* S_A over A is the set

$$S_A := \bigcup_{z\in A} W^{ss}(z) = \{(y,z) \in V^{ss} \times A\}.$$

Remark 3.1. In other terms, a stable tube is a bundle of strong stable manifolds over a coordinate patch in the unstable direction.

We fix a point $x_1 \in W \cap \mathcal{N}_K$, and a chart $\mathcal{V} \cong V^{ss} \times V^u \subset \mathcal{N}_K \cap W'$ as above. Let $A \subset V^u$ be a relatively compact open set. Then $S_A \subset W$, and hence $u|_{S_A} = 0$.

To exploit this vanishing microlocally, we define a test distribution $\rho_A \in \mathcal{D}'(U)$ supported on S_A, built via a leafwise disintegration of the volume measure. To construct smooth test distributions supported along stable tubes, we use the disintegration of the Liouville measure along stable leaves in the universal cover. The following classical result, which follows from [**PPS15**, Theorem 7.6] in our setting, provides the required regularity:

Theorem 3.2. *Let $\widetilde{M}$ be the universal cover of M, and let $S\widetilde{M}$ denote its unit tangent bundle. Then the Liouville measure $\tilde{\mu}$ on $S\widetilde{M}$ admits a disintegration along the strong stable foliation. That is, in any sufficiently small local product neighborhood of the form $V^{ss} \times V^u \subset S\widetilde{M}$, we can write*

$$d\tilde{\mu} = \rho(y,z)\, d\,\mathrm{vol}_{W^{ss}(z)}(y)\, dz,$$

where $W^{ss}(z)$ is the strong stable manifold through $z \in V^u$, and $\rho(y,z)$ is a conditional density which is smooth in the stable variable y, continuous in the transverse variable z, and locally uniformly bounded.

This smooth disintegration is equivariant under the action of $\pi_1(M)$, and hence descends to local product neighborhoods in SM. In particular, given a local product chart $\mathcal{V} \cong V^{ss} \times V^u \subset SM$ near a point in the trapped set K, we may define a distribution $\rho_A \in \mathcal{D}'(SM)$ supported on the associated stable tube $S_A := \bigcup_{z\in A} W^{ss}(z)$ (for a relatively compact open subset $A \subset V^u$) by

$$\langle \rho_A, f\rangle := \int_{z\in A} \left(\int_{y\in V^{ss}} f(y,z)\, \chi(y)\, \rho(y,z)\, dy \right) dz,$$

for all $f \in C_c^\infty(\mathcal{V})$, where $\chi \in C_c^\infty(V^{ss})$ is a fixed smooth bump function normalized so that $\int \chi = 1$.

This integral defines ρ_A as a compactly supported distribution in $\mathcal{D}'(SM)$. The key feature is that ρ_A is smooth along the strong stable direction. This ensures, via a standard integration-by-parts argument (cf. [**Wei17**, Proposition 6 and Theorem 7]), that the wavefront set of ρ_A is contained in the conormal bundle to the stable foliation:

$$\mathrm{WF}(\rho_A) \subset E_-^*.$$

To conclude the argument, we analyze the implications of the vanishing of u on the stable tube $S_A \subset \Gamma_+$. Since $\rho_A \in \mathcal{D}'(U)$ is supported in S_A, and $u|_{S_A} = 0$, we have $\operatorname{supp}(u) \cap \operatorname{supp}(\rho_A) = \emptyset$. Furthermore, as shown above, we have

$$\mathrm{WF}(\rho_A) \subset E_-^*, \quad \mathrm{WF}(u) \subset E_+^*, \quad \text{and} \quad E_+^* \cap E_-^* = \{0\}.$$

Hence, by Hörmander's criterion, it follows that the pairing $\langle u, \rho_A \rangle$ is well-defined. Since $\operatorname{supp}(u) \cap \operatorname{supp}(\rho_A) = \emptyset$, this pairing must vanish:

$$\langle u, \rho_A \rangle = 0.$$

We now propagate the vanishing of u across Γ_+. Let $x \in \Gamma_+$ be arbitrary. By definition, Γ_+ is forward-invariant under the flow, and its points have backward trajectories that remain in U. Since $K := \Gamma_+ \cap \Gamma_-$ is compact and $\Gamma_+ = W^u(K)$, the backward orbit $\{\varphi^{-t}(x)\}_{t \geq 0}$ accumulates on the trapped set K. In particular, there exists $t_0 \geq 0$ such that

$$x_0 := \varphi^{-t_0}(x) \in \mathcal{N}_K \cap \Gamma_+,$$

where $\mathcal{N}_K \subset U$ is the product neighborhood in which we constructed the stable tube S_A, and where u vanishes identically.

Since u is a generalized resonant state, the wavefront set $\mathrm{WF}(u)$ is invariant under the canonical lift Φ_t on T^*SM. In other terms, we have the identity

$$\mathrm{WF}(u) = \Phi_t(\mathrm{WF}(u)) \quad \text{for all } t \in \mathbb{R},$$

where Φ_t is the symplectic lift of φ_t (recall that this flow invariance of the wavefront set is a direct consequence of the distributional invariance under pullback).

Since u vanishes in a neighborhood of $x_0 = \varphi^{-t_0}(x)$, there exists a covector $\xi_0 \in E_+^*(x_0)$ and a conic neighborhood $V_0 \subset T^*SM$ of (x_0, ξ_0) such that

$$V_0 \cap \mathrm{WF}(u) = \emptyset.$$

Because $\mathrm{WF}(u)$ is invariant under the lifted flow Φ_t and $\mathrm{WF}(u) \subset E_+^*$ (which is backward-invariant), it follows that

$$(x, \xi) := \Phi_{t_0}(x_0, \xi_0) \notin \mathrm{WF}(u),$$

for some $\xi \in E_+^*(x)$.

Thus, ...

As $x \in \Gamma_+$ was arbitrary, it follows that u vanishes identically on all of Γ_+. This contradicts the assumption that u is a nontrivial generalized resonant state with support properly contained in Γ_+. We conclude that

$$\operatorname{supp}(u) = \Gamma_+,$$

which completes the proof of the first part of the theorem.

The argument for co-resonant states $v \in \mathcal{D}'(\mathcal{U})$, satisfying $(X^* + \bar{\lambda}_0)^j v = 0$, $\mathrm{WF}(v) \subset E_-^*$, proceeds identically upon reversing time and swapping stable and unstable directions, and yields:

$$\operatorname{supp}(v) = \Gamma_-.$$

This concludes the proof. □

4. Proof of Theorem 2.3

Proof. Let $R(\lambda) := (X + \lambda)^{-1}$ denote the meromorphic continuation of the resolvent of the geodesic vector field X on $\mathcal{U} \subset SM$. Fix a Pollicott–Ruelle resonance $\lambda_0 \in \mathbb{C}$. Then $R(\lambda)$ admits a Laurent expansion near $\lambda = \lambda_0$ of the form

$$R(\lambda; z, z') = \sum_{k=1}^{J} \frac{A_k(z, z')}{(\lambda - \lambda_0)^k} + R_{\mathrm{hol}}(\lambda; z, z'),$$

as shown in [**DG16**, Theorem 1] (see also the beginning of the proof of [**DG16**, Lemma 4.5]), where each $A_k \in \mathcal{D}'(\mathcal{U} \times \mathcal{U})$ is a finite-rank distribution, and $R_{\mathrm{hol}}(\lambda)$ is holomorphic near λ_0.

Let $\mathcal{R}_{\lambda_0}$ and $\mathcal{R}^*_{\lambda_0}$ denote the finite-dimensional spaces of generalized resonant and co-resonant states at λ_0:

$$\mathcal{R}_{\lambda_0} := \{u \in \mathcal{D}'(\mathcal{U}) \mid (X+\lambda_0)^m u = 0 \text{ for some } m \geq 1\},$$

$$\mathcal{R}^*_{\lambda_0} := \left\{v \in \mathcal{D}'(\mathcal{U}) \;\middle|\; (X^* + \overline{\lambda_0})^m v = 0 \text{ for some } m \geq 1\right\}.$$

Let $\{u_i\}_{i=1}^N$ and $\{v_j\}_{j=1}^N$ be dual bases of $\mathcal{R}_{\lambda_0}$ and $\mathcal{R}^*_{\lambda_0}$, normalized so that

$$\langle v_j, u_i\rangle = \delta_{ij}.$$

Then each A_k can be written in the form

$$A_k(z,z') = \sum_{i,j=1}^N \alpha_{ij}^{(k)} u_i(z) \otimes v_j(z'),$$

where $\alpha_{ij}^{(k)} \in \mathbb{C}$, and $N = \dim \mathcal{R}_{\lambda_0}$ denotes the dimension of the generalized resonant space associated to the resonance λ_0. By the results in [**DG16**], each u_i and v_j satisfy

$$\mathrm{WF}(u_i) \subset E_+^*, \quad \mathrm{supp}(u_i) \subset \Gamma_+, \qquad \mathrm{WF}(v_j) \subset E_-^*, \quad \mathrm{supp}(v_j) \subset \Gamma_-.$$

Therefore, the kernel A_k satisfies

$$\mathrm{WF}(A_k) \subset E_+^* \times E_-^*, \quad \mathrm{supp}(A_k) \subset \Gamma_- \times \Gamma_+.$$

Remark 4.1. The basis elements u_i and v_j used to express the Laurent coefficients A_k as sums of rank-one distributions correspond to generalized resonant and co-resonant states at the resonance λ_0. This representation parallels the spectral projectors Π_k in [**DG16**, Theorem 1], which appear in the Laurent expansion of the meromorphic continuation of the resolvent and act as finite-rank operators onto generalized resonant spaces. Each Π_k has a Schwartz kernel of the form

$$\Pi_k(z,z') = \sum_{i=1}^N u_i^{(k)}(z) \otimes v_i^{(k)}(z'),$$

where the $u_i^{(k)}$ are generalized resonant states and the $v_i^{(k)}$ are dual co-resonant states satisfying $(X^* + \overline{\lambda_0})^m v_i^{(k)} = 0$. In our setting, the A_k play the role of these kernels, and our expression

$$A_k(z,z') = \sum_{i,j=1}^N \alpha_{ij}^{(k)} u_i(z) \otimes v_j(z')$$

makes this structure explicit in terms of dual bases of the resonant and co-resonant spaces. The dual pairing $\langle v_j, u_i\rangle = \delta_{ij}$ reflects the action of these projection operators on test distributions.

Now consider the scattering operator

$$\mathcal{S}_g(\lambda)(y_-, y_+) := -(\iota_{\partial^-} \times \iota_{\partial^+})^* R(\lambda; y_-, y_+),$$

defined by the pullback of the resolvent kernel to the incoming and outgoing boundary components of $\mathcal{U}$, where $\iota_{\partial^\pm} : \partial_\pm \mathcal{U} \hookrightarrow \mathcal{U}$ are the canonical inclusion maps. The pullback is well-defined as a distribution by the transversality of $\mathrm{WF}(R(\lambda)) \subset E_+^* \times E_-^*$ to the conormal bundle of $\partial_- \mathcal{U} \times \partial_+ \mathcal{U}$, as proven in [**Cha24**, Corollary 6] in dimension 2 and [**CGL24**, Lemma 2.7] in general.

Define

$$S_k := -(\iota_{\partial^-} \times \iota_{\partial^+})^* A_k \in \mathcal{D}'(\partial_- \mathcal{U} \times \partial_+ \mathcal{U}).$$

To prove that $\mathcal{S}_g(\lambda)$ has a pole at λ_0, we must show that at least one S_k is nonzero. By Theorem 2.1, each u_i and v_j have full support on Γ_+ and Γ_-, respectively. Since $\Gamma_\pm$ consist of unit tangent vectors whose geodesics remain in $\mathcal{U}$ for all positive or negative time, respectively, and $\mathcal{U}$ is a compact neighborhood of the trapped set with strictly convex boundary, it follows that for every neighborhood $V \subset \mathcal{U}$ that is arbitrarily close to $\partial_\pm \mathcal{U}$, we have

$$\Gamma_\pm \cap V \neq \emptyset.$$

Therefore, it follows that the distributions u_i and v_j have nonvanishing topological support arbitrarily close to the boundary. Therefore, $\mathcal{S}_g(\lambda)$ has a pole at λ_0, i.e. it has a pole at λ_0 of order *at most* J, with Laurent expansion

$$\mathcal{S}_g(\lambda) = \sum_{k=1}^{J'} \frac{S_k}{(\lambda - \lambda_0)^k} + \text{holomorphic}, \qquad 1 \leq J' \leq J,$$

where at least one S_k is nonzero. This establishes the inclusion

$$\{\text{Pollicott–Ruelle resonances of } X\} \subset \{\text{poles of } \mathcal{S}_g(\lambda)\}.$$

Conversely, suppose λ_0 is a pole of $\mathcal{S}_g(\lambda)$. Since $\mathcal{S}_g(\lambda)$ is defined as a pullback of the meromorphic resolvent $R(\lambda)$, it follows that $R(\lambda)$ must also have a pole at λ_0. Therefore,

$$\{\text{poles of } \mathcal{S}_g(\lambda)\} \subset \{\text{Pollicott–Ruelle resonances of } X\}.$$

Combining both directions, we conclude:

$$\{\text{Pollicott–Ruelle resonances of } X\} = \{\text{poles of } \mathcal{S}_g(\lambda)\}.$$

□

References

[CGL24] Mihajlo Cekić, Colin Guillarmou, and Thibault Lefeuvre, *Local lens rigidity for manifolds of Anosov type*, Anal. PDE **17** (2024), no. 8, 2737–2795. MR 4810080

[Cha24] Yann Chaubet, *Closed geodesics with prescribed intersection numbers*, Geom. Topol. **28** (2024), no. 2, 701–758. MR 4718126

[DG16] Semyon Dyatlov and Colin Guillarmou, *Pollicott-Ruelle resonances for open systems*, Ann. Henri Poincaré **17** (2016), no. 11, 3089–3146. MR 3556517

[Gui17] Colin Guillarmou, *Lens rigidity for manifolds with hyperbolic trapped sets*, J. Amer. Math. Soc. **30** (2017), no. 2, 561–599. MR 3600043

[PPS15] Frédéric Paulin, Mark Pollicott, and Barbara Schapira, *Equilibrium states in negative curvature*, Astérisque (2015), no. 373, viii+281. MR 3444431

[Wei17] Tobias Weich, *On the support of Pollicott-Ruelle resonanant states for Anosov flows*, Ann. Henri Poincaré **18** (2017), no. 1, 37–52. MR 3592089

Institut für Mathematik, Universität Paderborn
Email address: hqs@umd.edu

Institut für Mathematik, Universität Paderborn
Email address: weich@math.upb.de