UNIQUE ERGODICITY AND DEVIATION FOR TRANSLATION FLOWS VIA MICROLOCAL ANALYSIS

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Abstract. We introduce a new method, based on renormalization and microlocal analysis, that recovers a number celebrated results in Teichmüller theory: a quantitative form of Masur's criterion [**Mas82**], first proved by [**For02**]. Our work extends the known results to include observables of *anisotropic* regularity. Work in progress, parts of which is incomplete in this version of the paper, treats two other problems: (1) a new proof of a result on the deviation of ergodic averages, first proved by Forni [**For02**] for strata, and for all g_t-invariant ergodic probability measures by Bufetov [**Buf14**], and (2) we establish a priori bounds for smooth extensions of solutions of the cohomological equation for generic translations flows with respect to any g_t-invariant ergodic probability measures, with the aim to control the anisotropic Sobolev norms of the solutions dynamically (i.e., via renormalization).

1. Introduction

[1]The moduli space of translation surfaces $\mathcal{H}_\kappa$ provides a natural framework for the renormalization of *generic* translation flows on Riemann surfaces. Each point in $\mathcal{H}_\kappa$ corresponds to a Riemann surface equipped with a holomorphic 1-form—also known as an abelian differential—which defines a flat metric with conical singularities. Away from the cone points, the structure allows one to define directional (or linear) flows on the surface, referred to as translation flows.

The Teichmüller geodesic flow is a dynamical system on $\mathcal{H}_\kappa$, but it also has a concrete geometric interpretation: it deforms a (degenerate) flat metric on a genus $g \geq 2$ Riemann surface via a quasiconformal map that stretches in the horizontal direction and contracts in the vertical direction. This deformation has a dynamical consequence—it effectively accelerates the horizontal trajectories of the translation flow, making long orbit segments appear shorter in the renormalized geometry.

Viewed from this perspective, the Teichmüller flow functions as a renormalizing dynamical system: instead of analyzing long trajectories of a translation flow on a fixed surface, one studies uniformly bounded-length segments of the flow on a family of surfaces evolving under the Teichmüller flow. In this way, the long-term behavior of a translation flow is encoded in the evolution of a single orbit in $\mathcal{H}_\kappa$.

A key insight, due to Masur [**Mas82**], is that if the Teichmüller flow orbit of a translation surface returns infinitely often to a compact subset of $\mathcal{H}_\kappa$, then the corresponding horizontal translation flow is uniquely ergodic. This criterion illustrates

[1]Last updated on April 25, 2025.

a deep connection between recurrence in moduli space and statistical properties of translation flows on surfaces.

This viewpoint transforms the study of statistical properties of generic translation flows—such as unique ergodicity, weak mixing, or deviations of ergodic averages—into questions about the dynamics of the Teichmüller flow and the geometry of the moduli space, equipped with the Teichmüller metric. In this context, a property is said to hold *generically* if it holds for almost every translation surface with respect to a g_t-invariant, ergodic probability measure whose support lies in $\mathcal{H}_\kappa$.

Our work uses tools from microlocal analysis to revisit several classical results in Teichmüller dynamics: the microlocal analytic frameworks allows us to define anisotropic Sobolev norm that are adapted to the flat structure, and renormalization allows us to control the boundedness of such norms dynamically. We emphasize that our approach is general and applies to a broad class of renormalization problems—including those outside of Teichmüller theory—that concern "generic" dynamical systems which are not fixed or periodic points of the renormalization dynamics.

2. Results

The purpose of this paper is introduce a new method, based on renormalization and microlocal analysis, that controls the deviation of ergodic averages for generic translation flows on translation surfaces, and to solve the cohomological equation for generic translation flows with respect to any g_t-invariant ergodic probability measure, where g_t denotes the Teichmüller geodesic flow.

The paper is divided into four parts.

The first part of the paper introduces and develops a renormalization cocycle over the moduli space of Abelian differentials $\mathcal{H}_\kappa$. An isotropic form of this cocycle was first introduced in the work of Forni **[For02]**, in his celebrated work on the deviation of ergodic averages, later extended by Bufetov **[Buf14]** to cover all g_t-invariant ergodic probability measures. Our paper introduces an *anisotropic* form of this renormalization cocycle (which we call *the Forni cocycle*, following R. Krikorian **[Kri03]** and W. Veech **[Vee08]**), establishes its quasi-compactness, and proves the simplicity of its top Lyapunov exponent. These results were inspired by the works of Forni **[For02]**, Faure–Roy–Sjöstrand **[FRS08]** and Faure–Gouëzel–Lanneau **[FGL19]**. More elaborately, the first part accomplishes the following steps:

(1) Constructs an anisotropic Sobolev bundle over moduli space.
(2) Proves quasi-compactness of the Forni cocycle
(3) Applies the ∞-dimensional multiplicative ergodic theorem to decompose the anisotropic Sobolev bundle over moduli space into finitely many g_t-invariant Oseledets subbundles, modulo a contracting subbundle that corresponds to the "essential" spectrum of the cocycle.
(4) Proves simplicity of the 0 Lyapunov exponent via propagation of singularities.
(5) Rules out the existence of Lyapunov exponents that exceed 0.

Using this setup, we address the first application of these methods, which is a quantitative form of Masur's criterion [**Mas82**], first proved by [**For02**] for smooth observables. Extending the latter result to anisotropic observables, our first result is therefore:

Theorem 2.1. *Assume that the forward Teichmüller orbit $g_{\mathbb{R}^+}(M,\omega)$ visits a compact set $K \subset \mathcal{H}_\kappa$ with positive frequency, that is,*

$$f_K := \liminf_{t\to+\infty} \ \mathrm{Leb}(\{t \geq 0 | g_t(M,\omega) \in K\}) > 0\,.$$

Then there exist constants $C(M,\omega) > 0$ and $\alpha > 0$ such that, for h, v sufficiently large, for all distributions $f \in W^{-v,h}(M)$ of zero average and for all $(p,T) \in M\times\mathbb{R}^+$, such that p has an infinite forward orbit under $\phi^X_{\mathbb{R}}$, we have

$$\big|\frac{1}{T}\int_0^T f\circ\phi^X_t(p)dt\big| \leq C(M,\omega)\|f\|_{\mathcal{W}^{-v,h}(M)}\, T^{-\alpha}\,.$$

Moreover, by the Poincare recurrence theorem, we can derive

Corollary 2.2. *Let ν be any g_t-invariant, ergodic probability measure on $\mathcal{H}_\kappa$. Then for ν-almost every translation surface $(M,\omega) \in \mathcal{H}_\kappa$, there exist constants $C(M,\omega) > 0$ and $\alpha > 0$ such that, for all h, v sufficiently large, all distributions $f \in W^{-v,h}(M)$ of zero average, and all $(p,T) \in M \times \mathbb{R}^+$ such that p has an infinite forward orbit under $\phi^X_{\mathbb{R}}$, we have:*

$$\left|\frac{1}{T}\int_0^T f\circ\phi^X_t(p)\,dt\right| \leq C(M,\omega)\,\|f\|_{W^{-v,h}(M)}\,T^{-\alpha}.$$

Informally, the space $W^{-v,h}(M)$, which is defined using the theory of variable-order pseudodifferential operators, consists of distributions that are of Sobolev regularity h along the horizontal direction, and of Sobolev regularity $-v$ along the vertical direction. Moreover, the regularity varies smoothly across phase space, interpolating between the horizontal and vertical directions via a smooth order function.

In order to prove this result, we first establish an anisotropic Sobolev restriction theorem for rectangles in Appendix B, and use it to prove an anisotropic version of a Sobolev trace theorem 7 that was first established in an isotropic form by Forni.

Work in progress, parts of which is incomplete in this version of the paper, uses the same framework discussed above to address deviation of ergodic averages (recovering the known results) and establishing smooth extensions of solutions of the cohomological equations for measure generic translation flows (going past what is known). For this purpose, an anisotropic Sobolev extention theorem is established in the appendix.

Structure of the paper. In Section 3, we review foundational background on translation surfaces, the moduli space $\mathcal{H}_\kappa$, the Teichmüller flow, and the associated horizontal and vertical vector fields. We also define smooth functions on translation surfaces and recall the the notion of variable-order pseudo-differential operators in our context. In Section 4, we construct the anisotropic Sobolev bundle, and define the anisotropic Forni cocycle and establish its boundedness and quasi-compactness on the constructed bundles. Section 5 applies the infinite-dimensional multiplicative ergodic theorem (MET) to obtain a measurable Oseledets splitting of the anisotropic bundle. Section 6 is devoted to proving the simplicity of the top Lyapunov exponent. In Section 7, we prove an anisotropic Sobolev trace theorem. Section 8 presents the first main application of the theory: a quantitative form of unique ergodicity for translation flows. In Appendix A, we prove a standard lemma of the order function used to construct our anisotropic Sobolev bundle. Finally, in Appendix B, we establish Sobolev restriction and extension theorems tailored to our anisotropic setting.

3. Preliminaries.

3.1. **Translation surfaces.** Let M be a Riemann surface of genus $g \geq 2$, and ω a holomorphic 1-form on S. The pair (M, ω) is called a *translation surface*, since ω induces a *translation atlas* whose coordinate changes are translations on $\mathbb{C} \equiv \mathbb{R}^2$. In other terms, ω gives a flat metric with finitely many conical singularities and trivial holonomy on S, and the zero set of ω characterizes the singularity set of the conical metric. The area of a translation surface is given by $\int_S \omega \wedge \overline{\omega}$. We will refer to the pair (S, ω) as just ω.

3.2. **Moduli Space.** Let $\mathcal{TH}_g$ be the Teichmüller space of unit-area translation surfaces of genus $g \geq 2$, and let $\mathcal{H}_g = \mathcal{TH}_g/\mathrm{Mod}_g$ be the corresponding moduli space, where Mod_g denotes the mapping class group. The space $\mathcal{H}_g$ is partitioned into strata $\mathcal{H}_\kappa$, which consist of all unit-area translation surfaces whose conical singularities have total angles $2\pi(1+\kappa_1), \ldots, 2\pi(1+\kappa_s)$, as $\kappa = (\kappa_1, \ldots, \kappa_s)$ varies over multi-indices with $\sum \kappa_i = 2g-2$.

Local period coordinates on each stratum are defined by the map which takes every holomorphic 1-form ω to its cohomology class $[\omega]$ in $H^1(S, \Sigma_\omega, \mathbb{C})$, relative to the set Σ_ω of its zeros. The set of all period coordinate maps defines an affine structure on each stratum, since all changes of coordinates are given by affine maps.

3.3. **The $\mathrm{SL}(2,\mathbb{R})$ Action.** The group $\mathrm{SL}(2,\mathbb{R})$ acts naturally on the space of translation surfaces by post-composing with charts, preserving both the flat structure and the area form the area form $dA_\omega := -\frac{i}{2}\,\omega \wedge \bar{\omega}$. This action descends to the moduli space and Teichmüller space of translation surfaces, and it preserves each stratum $\mathcal{H}_\kappa$. In this paper, we will primarily focus on the diagonal subgroup

$$g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad t \in \mathbb{R},$$

which generates the Teichmüller geodesic flow. Throughout, we will consider a g_t-invariant ergodic probability measure ν whose support lies in a stratum $\mathcal{H}_\kappa$.

We remark that the class of g_t-invariant measures is strictly larger than the class of $\mathrm{SL}(2,\mathbb{R})$-invariant measures. For example, every closed orbit of the Teichmüller flow, corresponding to a linear pseudo-Anosov map, supports a g_t-invariant ergodic probability measure that is not $\mathrm{SL}(2,\mathbb{R})$-invariant.

3.4. **Horizontal and vertical vector fields.** We define the horizontal and vertical vector fields X and Y via contraction with the real and imaginary parts of the holomorphic 1-form ω:

$$\iota_X \mathrm{Re}(\omega) = -\iota_Y \mathrm{Im}(\omega) = 1, \quad \text{and} \quad \iota_X \mathrm{Im}(\omega) = \iota_Y \mathrm{Re}(\omega) = 0.$$

In canonical coordinates $z = x + iy$, centered at a regular point (i.e., where $\omega = dz$), the area form becomes $dA_\omega = dx \wedge dy$, and the vector fields take the familiar form:

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y}.$$

Near a cone point of angle $2\pi(k+1)$, with local coordinate z such that $\omega = z^k dz$, the vector fields become:

$$X = |z|^{-2k}\left(\mathrm{Re}(z^k)\frac{\partial}{\partial x} - \mathrm{Im}(z^k)\frac{\partial}{\partial y}\right), \quad Y = |z|^{-2k}\left(\mathrm{Im}(z^k)\frac{\partial}{\partial x} + \mathrm{Re}(z^k)\frac{\partial}{\partial y}\right).$$

To resolve the singularity at the cone points, we use the branched covering map:

$$\pi_k(z) = \frac{z^{k+1}}{k+1},$$

which defines a $(k+1)$-fold branched cover $\pi_k : U(p) \to D \subset \mathbb{C}$ around each singularity $p \in \Sigma$, satisfying $\pi_k^*(dz) = \omega$ and:

$$(\pi_k)_* X = \frac{\partial}{\partial x}, \quad (\pi_k)_* Y = \frac{\partial}{\partial y}.$$

3.5. **Trivialization of the Tangent and Cotangent Bundles.** The tangent bundle TM and the cotangent bundle T^*M can be globally trivialized over $M \setminus \Sigma$, since both admit nowhere vanishing global frames. Specifically, the canonical horizontal and vertical vector fields X_ω and Y_ω—defined respectively on the horizontal and vertical foliations $\ker(\mathrm{Im}\,\omega)$ and $\ker(\mathrm{Re}\,\omega)$—form an orthonormal basis of the tangent space at every nonsingular point:

$$T_m(M) = \mathrm{span}\{X_\omega, Y_\omega\}, \quad \text{for all } m \in M \setminus \Sigma.$$

Dually, the real and imaginary parts of the Abelian differential ω form an orthonormal basis of the cotangent space:

$$T_m^*(M) = \mathrm{span}\{\mathrm{Re}\,\omega, \mathrm{Im}\,\omega\} = \mathrm{span}\{X_\omega^*, Y_\omega^*\},$$

where we set

$$X_\omega^* = \mathrm{Re}\,\omega, \qquad Y_\omega^* = \mathrm{Im}\,\omega.$$

In particular, we have the decompositions

$$T(M \setminus \Sigma) = \mathbb{R}X_\omega \oplus \mathbb{R}Y_\omega \equiv M \setminus \Sigma \times \mathbb{R}^2, \tag{3.1}$$

$$T^*(M \setminus \Sigma) = \mathbb{R}X_\omega^* \oplus \mathbb{R}Y_\omega^* \equiv M \setminus \Sigma \times \mathbb{R}^2. \tag{3.2}$$

The action of the Teichmüller flow $\{g_t\} \subset \mathrm{SL}(2, \mathbb{R})$ preserves this structure. Indeed, it acts linearly on the basis $\{\mathrm{Re}\,\omega, \mathrm{Im}\,\omega\}$ via

$$g_t\,\omega = e^t \mathrm{Re}\,\omega + i\,e^{-t} \mathrm{Im}\,\omega,$$

from which we obtain:

$$X_{g_t\omega} = g_t X_\omega = e^{-t} X_\omega, \qquad Y_{g_t\omega} = g_t Y_\omega = e^{t} Y_\omega,$$
$$X^*_{g_t\omega} = g_t X^*_\omega = e^{t} X^*_\omega, \qquad Y^*_{g_t\omega} = g_t Y^*_\omega = e^{-t} Y^*_\omega.$$

Thus, the decompositions (3.1) and (3.2) are preserved under the Teichmüller flow, with the g_t-action scaling the basis vectors accordingly.

3.6. **Smooth functions on translation surfaces.** Let $\Sigma \subset M$ be the set of cone points of the translation surface (M, ω). For each $p \in \Sigma$ with cone angle $2\pi(k+1)$, fix a neighborhood $U(p) \subset M$ and a local branched covering chart

$$\pi_k : D \to U(p), \quad \pi_k(w) = \frac{w^{k+1}}{k+1},$$

where $D \subset \mathbb{C}$ is a disk centered at the origin. This map resolves the singularity at p: $\omega = \pi_k^*(dw)$. Following **[For02]**, we define the space $C^\infty_\omega(M)$ of *tempered* smooth functions adapted to the translation structure as follows:

Definition 3.1. A function $f : M \to \mathbb{R}$ belongs to $C^\infty_\omega(M)$ if, for every chart (U, π_U) in a translation atlas $\mathcal{U}$ on M we have

$$f|_U \in (\pi_U)^* C^\infty(\pi_U(U)).$$

Next, we turn to smooth functions defined on the cotangent bundle. The map π_k lifts naturally to a map on the cotangent bundle,

$$\widetilde{\pi}_k : T^*D \to T^*U(p),$$

defined in coordinates as follows: write $w = u + iv$ and let (η_u, η_v) be the fiber (covector) coordinates on T^*D, and (ξ_x, ξ_y) the fiber coordinates on $T^*U(p)$. Then

$$(x, y) = \pi_k(u, v), \quad (\xi_x, \xi_y) = (d\pi_k)^*(\eta_u, \eta_v),$$

where $(d\pi_k)^*$ is the transpose of the Jacobian matrix $d\pi_k$ of the base map. This map pulls back covectors from $T^*U(p)$ to T^*D, ensuring that cotangent vectors transform appropriately under change of coordinates. We can now similarly define the space $C^\infty(T^*M)$ of *tempered* smooth functions on the cotangent bundle, adapted to the translation structure, as follows:

Definition 3.2. A function $a : T^*M \to \mathbb{R}$ belongs to $C^\infty_\omega(T^*M)$ if, for every chart (U, π_U) in a translation atlas $\mathcal{U}$ on M, the restriction satisfies

$$a|_{T^*U} \in \tilde{\pi}_U^* C^\infty(T^*\pi_U(U)),$$

where $\tilde{\pi}_U$ is the canonical lift of π_U to the cotangent bundle.

3.7. **Variable-Order Pseudo-Differential Operators.** We now introduce *variable-order* pseudo-differential operators. Our treatment follows the approach of [**FRS08**, Appendix A], adapted to the local structure induced by the translation atlas. Thus, let us fix a translation atlas $\{U_i, \phi_i\}_{i \in I}$ of the translation surface M such that every $U_i \subset M$ contains at most one singular point. Denote by $V_i =: \phi_i(U_i) \subset \mathbb{R}^2$ and by V an open, bounded, connected subset of $\mathbb{R}^2$. For $\xi \in \mathbb{R}^2$, define $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$, where $|\xi|$ is the Euclidean norm of ξ in $\mathbb{R}^2$. We recall the definition of order function.

Definition 3.3. A smooth function $m \in C^\infty(V \times \mathbb{R}^2)$ is an *order function*, i.e., $m \in S^0(T^*V)$, if

(1) for every compact set $K \subset V$, for every multi-indeces $\alpha, \beta \in \mathbb{N}^2$ and any $(x, \xi) \in T^*V \equiv V \times \mathbb{R}^2$
$$|\partial_\xi^\alpha \partial_x^\beta m(x, \xi)| \le C_{K,\alpha,\beta} \langle \xi \rangle^{-|\alpha|},$$
for some uniform constants $C_{K,\alpha,\beta} > 0$;

(2) $\sup_{\xi \in \mathbb{R}^2} |m(x, \xi)|$ is uniformly bounded in V.

A smooth function $m \in C^\infty(T^*M)$ is an *order function*, i.e., $m \in S^0(T^*M)$, if $m(\phi_i(x), \phi_{i*}(\xi)) \in S^0(T^*V_i)$, for every $i \in I$.

Definition 3.4. Let $\rho \in (\frac{1}{2}, 1]$. A smooth function $a \in C^\infty(T^*V)$ is a *symbol of variable order* $m \in S^0(T^*V)$, i.e., $a \in S_\rho^m(x,\xi)(T^*V)$, if for every compact set $K \subset V$, for every multi-indeces $\alpha, \beta \in \mathbb{N}^2$ and any $(x, \xi) \in T^*V \equiv V \times \mathbb{R}^2$
$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \le C_{K,\alpha,\beta} \langle \xi \rangle^{m(x,\xi) - \rho|\alpha| + (1-\rho)|\beta|}$$
for some constants $C_{K,\alpha,\beta} > 0$.

A smooth function $a \in C^\infty(T^*M)$ is an *symbol of variable order* $m \in S^0(T^*M)$, i.e., $a \in S_\rho^m(x,\xi)(T^*M)$, if $a(\phi_i(x), \phi_{i*}(\xi)) \in S_\rho^m(x,\xi)(T^*V_i)$, for every $i \in I$.

Remark 3.5. Notice that the order function m in Definition 3.3 is nothing but a symbol of constant order 0 and parameter $\rho = 1$.

We can now introduce the pseudodifferential operators with variable order.

Definition 3.6. A is a PDO of variable order m on $C^\infty(V)$ if it has the following form
$$Au(x) = \frac{1}{(2\pi)^2} \int_V \int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi} a(x, \xi) u(y) d\xi dy, \tag{3.1}$$
where $a(x, \xi)$ is a symbol of variable order $m(x, \xi)$. A is referred to as the *left quantization* of the symbol a. Alternatively, one writes $A = \mathrm{Op}(a(x, \xi))$.

Remark 3.7. The definition of PDOs for smooth manifolds depends on the choice of coordinate charts. On the other hand, there is a well-defined notion of *principal symbol* which is independent of the choice of charts. We refer the reader to [**FRS08**, Section A.1.3] for a detailed discussion of the topic.

In (3.1), dy locally represents the area form $\frac{i}{2}\omega \wedge \bar{\omega} = \mathrm{Re}\omega \wedge \mathrm{Im}\omega$, that is the Lebesgue measure associated with a basis of T_mM. Similarly, $d\xi := d\xi_x \wedge d\xi_y$ denotes the (local) area form on T^*M. In fact, it is given by the symplectic area element $\sigma^2/2$ associated to the canonical symplectic 2-form:
$$\sigma := d\alpha = \mathrm{Re}\omega \wedge d\xi_x + \mathrm{Im}\omega \wedge d\xi_y,$$
where
$$\alpha = \xi_x \mathrm{Re}\omega + \xi_y \mathrm{Im}\omega,$$
and we refer to [**Zwo22**, Page 342] for more details.

4. The Forni Cocycle

In **[For02]**, Forni introduced the following construction: for any $\omega \in \mathcal{TH}(\kappa)$, let $C^{-\infty}(M,\omega)$ denote the space of tempered distributions on M, and define the following trivial product bundle

$$\mathcal{T}C^{-\infty}_{\kappa}(M) = \{(\omega, D) | D \in C^{-\infty}_{\omega}(M,\omega)\}$$

over $\mathcal{TH}(\kappa)$. Forni extends the $\mathrm{SL}(2,\mathbb{R})$-action by parallel transport via the trivial connection on this bundle as follows: for $A \in \mathrm{SL}(2,\mathbb{R})$, we have

$$A(\omega, D) = (A \cdot \omega, D)$$

To pass to the quotient, let us note that the diagonal action of $\mathrm{Diff}^+(M)$ on $\mathcal{T}C^{-\infty}_{\kappa}(M)$ is as follows: for any orientation-preserving diffeomorphism $\phi \in \mathrm{Diff}^+(S)$, we have

$$\phi(\omega, D) = (\phi^*(\omega), \phi_*(D)),$$

so that ϕ acts on ω by pullback and on D by pushforward. Then observes that the $\mathrm{SL}(2,\mathbb{R})$-action commutes with the $\mathrm{Diff}^+(M)$-action, which further implies that the $\mathrm{SL}(2,\mathbb{R})$-action is well-defined on the quotient bundle

$$C^{-\infty}_{\kappa}(M) := \mathcal{T}C^{-\infty}_{\kappa}(M)/\mathrm{Diff}^+(M).$$

The Forni cocycle is defined to be the lift of the $\{g_t\}$-action to the bundle $C^{-\infty}_{\kappa}(M)$, by parallel transport with respect to the trivial connection.

Remark 4.1. In **[For02]**, Forni referred to this cocycle as the *renormalization cocycle*, or *transfer cocycle*. Subsequently, R. Krikorian **[Kri03]** and W. Veech **[Vee08]** have both named this cocycle after Forni, and we continue to adopt their terminology in this paper.

The following lemma introduces the order function m and the subsequent symbol a_m. We anticipate that the a_m, also called escape function, satisfies an additional property: it is decreasing along the orbits of the geodesic flow. This feature is crucial in proving that the Forni cocycle (4.1) is quasicompact (see Theorem 4.6).

Lemma 4.2. *Let $v \in \mathbb{R}^+$ and $h \in \mathbb{R}^+$. There exists an order function $m_\omega \in S^0_1 \subset C^\infty_\omega(T^*M)$, with values in the interval $[-h, v]$, that fulfills the following properties. For all $\xi \in \mathbb{R}^2 \setminus \{0\}$, with $|\xi| \geq 1$, m is defined projectively, i.e., it only depends on the direction $\xi/|\xi|$. Moreover, for all $\xi \in \mathbb{R}^2$, $m(\xi) \equiv v$ in a conical neighborhood of $Re(\omega)$, and $m(\xi) \equiv -h$ in a conical neighborhood of $Im(\omega)$, for $|\xi| > 1$. In addition,*

$$\forall t > 0,\ m_{g_t\omega}(\xi) - m_\omega(\xi) \leq 0. \tag{4.1}$$

Let us define

$$a^\omega_m(\xi) := \langle \xi \rangle^{m_\omega(\xi)},$$

where $\langle \xi \rangle = \sqrt{1+|\xi|^2}$. a_m^ω is a symbol of class $S_\rho^{m_\omega(\xi)}$ and it satisfies the following: for all $t>0$, there exists $R_t \in \mathbb{R}^+$, depending on t, such that

$$\frac{a_m^{g_t\omega}(\xi)}{a_m^\omega(\xi)} < Ce^{-\frac{1}{2}\min\{v,h\}t} \tag{4.2}$$

for $|\xi| > R_t$ and a uniform constant $C>0$.

The proof of this lemma is modeled after that in **[FRS08]**, and is deferred to Appendix A. Before defining an anisotropic Sobolev bundle, we define an anisotropic Sobolev space as follows

$$W_\omega^{v,-h}(M) := (\mathrm{Op}(a_m(x,\xi))^{-1}L^2(M,\omega),$$

for some $h \in \mathbb{R}^+$ and $v \in \mathbb{R}^+$, and we assign an anisotropic Sobolev norm to a distribution $D \in W_\omega^{v,-h}(M)$ as

$$\|D\|_{v,-h,\omega} := \| \mathrm{Op}(a_m(x,\xi))D\|_{L^2(\omega)}.$$

Equivalently, we can also define $W_\omega^{v,-h}(M)$ as

$$W_\omega^{v,-h}(M) := \{D \in C^{-\infty}(M,\omega) \colon \| \mathrm{Op}(a_m(x,\xi))D\|_{L^2(M,\omega)} < \infty\}.$$

The equivalent formulation has the following advantage: since

$$L^2(M,\omega) = \phi^* L^2(\phi^*(M,\omega)),$$

for all $\phi \in \mathrm{Diff}^+(M)$, this immediately shows that

$$W_\kappa^{v,-h}(M) := \{(D,\omega) \in C^{-\infty}(M,\omega) \colon \| \mathrm{Op}_\omega(a_m(x,\xi))D\|_{L^2(M,\omega)} < \infty\}/\mathrm{Diff}^+(M)$$

is a well-defined sub-bundle of $C_\kappa^{-\infty}(M)$. Moreover, since

$$L^2(g_t(M,\omega)) = L^2(M,\omega),$$

this implies that the *anisotropic* Forni cocycle, that is, the lift of the $\{g_t\}$-action to $W_\kappa^{v,-h}(M)$, is well-defined as an unbounded operator-valued cocycle $\mathcal{F}_t(\omega)$ on $C_\kappa^{-\infty}(M)$. Here, we use the terminology unbounded to mean not necessarily bounded. Unbounded operators require one to restrict to the domain of definition, which motivates the following considerations.

4.1. **Domain of definition.** Restricted to its domain of definition, the (anisotropic) Forni cocycle $\mathcal{F}_t(\omega)$ is the *identity* cocycle, in the following sense

$$\begin{aligned} \mathcal{F}_t(\omega) : \mathrm{Dom}(\mathcal{F}_t((\omega)) \subset W_\omega^{v,-h} &\to W_{g_t\omega}^{v,-h} \\ (\omega, D) &\to (g_t\omega, D), \end{aligned}$$

where

$$\begin{aligned} \mathrm{Dom}(\mathcal{F}_t(\omega)) &= \{D \in W_\omega^{v,-h} \colon D \in W_{g_t\omega}^{v,-h}\} \\ &= \{D \in C_\omega^{-\infty}(M) \colon D \in W_\omega^{v,-h} \text{ and } D \in W_{g_t\omega}^{v,-h}\}, \end{aligned}$$

Equivalently, the Forni cocycle $\mathcal{F}_t(\omega)$ is the operator-valued cocycle

$$\mathcal{F}_t(\omega) := \mathrm{Op}(a_m(x,\xi))^{-1}\,\mathrm{Op}(a_m(g_t(x,\xi))) : \mathrm{Dom}(\mathcal{F}_t(\omega)) \to W_\omega^{v,-h}, \tag{4.1}$$

where

$$\begin{aligned}\mathrm{Dom}(\mathcal{F}_t(\omega)) &= \{D \in W^{v,-h}_\omega \colon \mathcal{F}_t(\omega)D \in W^{v,-h}_\omega\}\\ &= \{D \in C^{-\infty}_\omega(M)\colon D \in W^{v,-h}_\omega \text{ and } \mathcal{F}_t((\omega)D \in W^{v,-h}_\omega\}\\ &= \{D \in C^{-\infty}_\omega(M)\colon \mathrm{Op}(a_m(x,\xi))D \& \mathrm{Op}(a_m(g_t(x,\xi)))D \in L^2(M,\omega)\}\end{aligned}$$

In fact, and relying crucially on the fact that $L^2(M,\omega) = L^2(g_t(M,\omega))$, we have just proved

Lemma 4.3. *The formulation in 4.1 identifies the fibers $W^{v,-h}_\omega$ and $W^{v,-h}_{g_t\omega}$, and thus defines a measurable trivialization of the bundle $W^{v,-h}_\kappa(M)$ when restricted to the orbit $\{g_t\omega\}$ and, more generally, to the support of any ergodic g_t-invariant probability measure.*

This lemma will be crucial for the rest of our microlocal considerations, especially in the application of the infinite-dimensional multiplicative ergodic theorem.

Remark 4.4. We refer to $\mathcal{F}_t(\omega)$ as a cocycle since, for $s, t \in \mathbb{R}$, $\mathcal{F}_{t+s}(\omega) = \mathcal{F}_s(g_t\omega)\mathcal{F}_t(\omega)$. That is, it is a multiplicative cocycle in the sense of dynamical systems. Moreover, the cocycle $\mathcal{F}_t(\omega)$ maps each $D \in \mathrm{Dom}(\mathcal{F}_t(\omega))$ to $\mathcal{F}_t(\omega)D \in \mathcal{W}^{v,-h}_\omega$.

Remark 4.5. It can be shown by standard arguments that $(\mathrm{Dom}(\mathcal{F}_t(\omega)), \|\cdot\|_{\mathrm{Dom},g_t\omega})$ and $(W^{v,-h}_\omega, \|\cdot\|_{v,-h,\omega})$ are Hilbert spaces, where $\|\cdot\|_{\mathrm{Dom},g_t\omega} := \|\cdot\|_{v,-h,\omega} + \|f\|_{v,-h,g_t\omega}$. Moreover, $\mathrm{Dom}(\mathcal{F}_t(\omega))$ is a densely defined subspace of $(W^{v,-h}_\omega, \|\cdot\|_{v,-h,\omega})$, and extends to a bounded operator from $W^{v,-h}_\omega$ to itself.

4.2. **Quasi-compactness of the Forni cocycle.** Restricted to its domain of definition, the Forni cocycle is a bounded operator-valued-cocycle from $W^{v,-h}_\omega$ to $W^{v,-h}_{g_t\omega}$. To establish our quasi-compactness estimates, where we shall appeal to the L^2 continuity of pseudo-differential operators of order 0, it will be useful to conjugate the Forni cocycle to one that is a bounded operator-valued-cocycle from $L^2(M,\omega)$ to $L^2(g_t(M,\omega)) = L^2(M,\omega)$. First, observe that

$$(\mathrm{Op}(a_m(x,\xi))W^{v,-h}_\omega = L^2(M,\omega),$$

and let

$$\begin{aligned}\mathcal{F}^{L2}_t(\omega) &:= (\mathrm{Op}(a_m(x,\xi))\,\mathcal{F}_t(\omega)\,(\mathrm{Op}(a_m(x,\xi))^{-1}\\ &= \mathrm{Op}(a_m \circ g_t(x,\xi))\,\mathrm{Op}(a_m(x,\xi))^{-1}.\end{aligned}$$

The L^2 Forni cocycle is therefore the conjugated Forni cocycle, as follows:

$$\begin{aligned}\mathcal{F}^{L2}_t(\omega) : \mathrm{Dom}(\mathcal{F}^{L2}_t((\omega)) \subset L^2(M,\omega) &\to L^2(M,\omega)\\ (\omega, D) &\to (g_t\omega, \mathcal{F}^{L2}_t(\omega)D),\end{aligned}$$

where

$$\begin{aligned}\mathrm{Dom}(\mathcal{F}^{L2}_t(\omega)) &:= \{D \in L^2(M,\omega)\colon \mathcal{F}^{L2}_t(\omega)D \in L^2(M,\omega)\}\\ &= \{D \in C^{-\infty}_\omega(M)\colon D \in L^2(M,\omega) \text{ and } \mathcal{F}^{L2}_t(\omega)D \in L^2(M,\omega)\}.\end{aligned}$$

We will now show

Theorem 4.6. *The densely-defined Forni cocycle*

$$\mathcal{F}_t(\omega) : Dom(\mathcal{F}_t(\omega)) \to W^{v,-h}_\omega$$

forms a family in $t > 0$ of quasi-compact operators with respect to the $\| \cdot \|_{v,-h,\omega}$ norm. Equivalently, The densely-defined L^2 Forni cocycle

$$\mathcal{F}^{L2}_t(\omega) : Dom(\mathcal{F}^{L2}_t(\omega)) \to L^2(M, \omega)$$

forms a family in $t > 0$ of quasi-compact operators with respect to the $\| \cdot \|_{L^2}$ norm: that is, it is a bounded cocycle with respect to the $\| \cdot \|_{L^2}$ norm, and can be written as

$$\mathcal{F}^{L2}_t(\omega) = \hat{r}_t(\omega) + \hat{k}_t(\omega) : Dom(\mathcal{F}^{L2}_t(\omega)) \to L^2(M, \omega)$$

with

$$\|\mathcal{F}^{L2}_t(\omega)\|_{L^2\to L^2} = \|\hat{r}_t\|_{L^2\to L^2} + \|\hat{k}_t\|_{L^2\to L^2},$$

where $\hat{r}_t$ is a compact cocycle, and $\|\hat{k}_t\|_{L^2\to L^2} \le Ce^{-\frac{1}{2}\min\{v,h\}t}$ for some $C := C_{v,h} > 0$.

Proof. The proof follows that of [**FRS08**, Theorem 1], and we note that we do not apply (nor need) Egorov's lemma in our setting. By the composition theorem for PDOs [**FRS08**, Theorem 5], and for any $t > 0$ and any ω, $\mathcal{F}^{L2}_t(\omega)$ is a (variable order) pseudo-differential operator in $\Psi^{m(g_t(x,\xi))-m(x,\xi)}_\rho$ whose symbol is $\frac{a_m \circ g_t}{a_m}$, modulo subleading corrections in $S^{m(g_t(x,\xi))-m(x,\xi)-(2\rho-1)}_\rho$. By (3.1), we have that $\mathcal{F}^{L2}_t(\omega) \in \Psi^0_\rho$, while (3.2) ensures that

$$\limsup \mathcal{F}^{L2}_t(\omega) \le Ce^{-\frac{1}{2}\min\{v,h\}t},$$

for $C := C_{v,h} = e^{\frac{1}{2}\min\{v,h\}\ln 4} > 0$. Therefore, by L^2 continuity for 0 order PDOs [FRS08, Lemma 14], we have that, for any $\epsilon > 0$,

$$\mathcal{F}^{L2}_t(\omega) = \hat{r}_{\epsilon,t}(\omega) + \hat{k}_{\epsilon,t}(\omega),$$

where $\hat{r}_{\epsilon,t}$ is a smoothing compact cocycle, and

$$\|\hat{k}_{\epsilon,t}\|_{L^2\to L^2} \le Ce^{-\frac{1}{2}\min\{v,h\}t} + \epsilon.$$

In fact, the proof of [FRS08, Lemma 14] shows that for any $\epsilon > 0$,

$$\|\mathcal{F}^{L2}_t(\omega)\|_{L^2\to L^2} = \|\hat{r}_{t,\epsilon}(\omega)\|_{L^2\to L^2} + \|\hat{k}_{t,\epsilon}(\omega)\|_{L^2\to L^2},$$

and this completes the proof, since the cocycles $\mathcal{F}^{L2}_t(\omega)$ and $\mathcal{F}_t(\omega)$ are unitarily equivalent and $\|\mathcal{F}_t(\omega)\|_{W^{v,-h}\to W^{v,-h}} = \|\mathcal{F}^{L2}_t(\omega)\|_{L^2\to L^2}$ □

Following [FRS08], define

$$W^{-v,h}_\omega := \mathrm{Op}(a_m(x,\xi))L^2(M, \omega),$$

and bserve that we can also define *the dual Forni cocycle* $\check{\mathcal{F}}_t(\omega)$, acting on the dual anisotropic Sobolev space $W^{-v,h}_\omega$, by

$$\check{\mathcal{F}}_t(\omega) := \mathrm{Op}(a_m(x,\xi))\ \mathrm{Op}(a_m(g_t(x,\xi))^{-1}.$$

Let us recall from [FRS08] that $W^{-v,h}$ and $W^{v,-h}$ are dual in the following: if $\phi \in W^{v,-h}$ and $\psi \in W^{-v,h}$,

$$\langle \phi, \psi \rangle_{W^{v,-h} \times W^{-v,h}} := (\mathrm{Op}(a_m(x,\xi))\phi, \mathrm{Op}(a_m(x,\xi))^{-1}\psi)_{L^2(M,\omega)}.$$

From which we can derive

$$\langle \phi, \psi \rangle_{W^{v,-h} \times W^{-v,h}} | \leq \|\phi\|_{W^{v,-h}} \|\psi\|_{W^{-v,h}}$$

Moreover, if $\phi \in W^{v,-h} \cap L^2(M,\omega)$ and $\psi \in W^{-v,h} \cap L^2(M,\omega)$,

$$\langle \phi, \psi \rangle_{W^{v,-h} \times W^{-v,h}} = \langle \phi, \psi \rangle_{L^2(M,\omega)}.$$

Let us record an invariance property of the distributional pairing under the action of these two cocycles. Specifically, for all $f \in W^{v,-h}_{g_t\omega}$ and $g \in W^{-v,h}_{g_{-t}\omega}$, the pairing satisfies

$$\langle f, g \rangle_{W^{v,-h}_{g_t\omega} \times W^{-v,h}_{g_{-t}\omega}} = \langle \mathcal{F}_t(\omega) f, \check{\mathcal{F}}_{-t}(\omega) g \rangle_{W^{v,-h}_{\omega} \times W^{-v,h}_{\omega}}.$$

By duality of $\check{\mathcal{F}}_{-t}(\omega)$ with respect to the distributional pairing$\langle \cdot, \cdot \rangle_{W^{v,-h} \times W^{-v,h}}$, we obtain the adjoint as follow

$$\langle \mathcal{F}_t(\omega) f, \check{\mathcal{F}}_{-t}(\omega) g \rangle_{W^{v,-h}_{\omega} \times W^{-v,h}_{\omega}} = \langle (\check{\mathcal{F}}_{-t}(\omega))^* \mathcal{F}_t(\omega) f, g \rangle_{W^{v,-h}_{\omega} \times W^{-v,h}_{\omega}}$$

5. Oseledets decomposition of the Anisotropic Sobolev Bundle

Let ν be any ergodic g_t-invariant probability measure, and let $\operatorname{supp} \nu \subset \mathcal{H}(\kappa)$ be its full measure set of Birkhoff-regular points. For ν-a.e. Birkhoff-regular ω, let $W^{v,-h}_{\omega}$ be the fiber over $\omega \in \operatorname{supp} \nu$ in $W^{v,-h}_{\kappa}(M)$. We will need the following

Lemma 5.1. *The Forni cocycle $\mathcal{F}_t(\omega)$ forms a family in $t > 0$ of* strongly measurable *operators on $W^{v,-h}_{\omega}$: that is, for any $D \in W^{v,-h}_{\omega}$, we have that $\omega \to \mathcal{F}_t(\omega) D$ is $(\Sigma_{supp\,\nu}, \mathcal{B}_{\Psi^{-m} \times \Psi^m})$-measurable, where $\Sigma_{supp\,\nu}$ is the σ-algebra on $supp\,\nu$ and $\mathcal{B}_{S^{-m}_{\omega} \times S^m_{\omega}}$ is the Borel σ-algebra (with respect to the strong operator topology) on $S^{-m}_{\omega} \times S^m_{\omega}$.*

Proof. For any $D \in \mathrm{Dom}(\mathcal{F}_t(\omega))$, for $t > 0$, we need to show that the map

$$\operatorname{supp}(\nu) \ni \omega \mapsto \mathcal{F}_t(\omega) D = \mathrm{Op}(a_m(x,\xi))^{-1} \,\mathrm{Op}(a_m(g_t(x,\xi)) D \in (\Psi^{-m} \times \Psi^m)(D)$$

is $(\Sigma_{\operatorname{supp}\nu}, \mathcal{B}_{S^{-m}_{\omega} \times S^m_{\omega}})$-measurable. We can do this in two stages. First, for each $\omega \in \operatorname{supp} \nu$, the map

$$S^{-m}_{\omega} \times S^m_{\omega} \subset C^{\infty}(T^*(M)) \times C^{\infty}(T^*(M)) \to \Psi^{-m}_{\omega} \times \Psi^m_{\omega}$$

is continuous with respect to the product Frechet topology on $\Psi^{-m}_{\omega} \times \Psi^m_{\omega}$, induced by the seminorms. On the other hand, by construction, the map

$$\operatorname{supp} \nu \ni \omega \mapsto C^{\infty}_{\omega}(T^*(M)) \times C^{\infty}_{\omega}(T^*(M))$$

is a $(\mathcal{F}_{\operatorname{supp}\nu}, \mathcal{B}_{C^{\infty}_{\omega}(T^*(M)) \times C^{\infty}_{\omega}(T^*(M))})$-measurable map from the measure space

$$(\operatorname{supp} \nu, \Sigma_{\operatorname{supp}\nu})$$

to the measure space

$$(C^{\infty}_{\omega}(T^*(M)) \times C^{\infty}_{\omega}(T^*(M)), \mathcal{B}_{C^{\infty}_{\omega}(T^*(M)) \times C^{\infty}_{\omega}(T^*(M))}).$$

This completes the proof of the lemma. □

Typically, one studies the spectral radius of a single operator. For the Forni cocycle, which parameterizes a family of operators, the (logarithmic) analogue of the spectral radius is the *maximal Lyapunov exponent*:

$$\lambda_1(\omega) := \lim_{t\to\infty} \frac{1}{t} \log \|\mathcal{F}_t(\omega)\|_{W^{v,-h}\to W^{v,-h}}.$$

Moreover, if one assumes

$$\log^+ \sup_{t\in[0,1]} (\|\mathcal{F}_t(\omega)\|_{v,-h,\omega}),\ \log^+ \sup_{t\in[0,1]} (\|\mathcal{F}_{1-t}(g_{1-t}\omega)\|_{v,-h,\omega}) \in L^1(X,\nu). \tag{5.1}$$

then the Kingman sub-additive ergodic theorem, together with ergodicity of the g_t-action, implies that $\lambda_1(\omega)$ is identically a constant for ν-a.e. ω, and so one writes λ_1.

Lemma 5.2. *Condition 5.1 holds.*

Proof. For any fixed $t \in [0,1]$, the operator norms $\|\mathcal{F}_t(\omega)\|_{v,-h,\omega}$ and $\|\mathcal{F}_{1-t}(g_{1-t}\omega)\|_{v,-h,\omega}$ are bounded, giving us the conclusion. □

We will also need to define a cocycle analogue of the (logarithm) of the essential spectral radius of a single operator. We first define

Definition 5.3. The Kuratowski measure of non-compactness of the (bounded) cocycle $\mathcal{F}_t(\omega) : W^{v,-h}_\omega \to W^{v,-h}_\omega$ is

$$\|\mathcal{F}_t(\omega)\|_{ic(X)} = \inf\{r > 0 : \mathcal{F}_t(\omega)(B) \text{ can be covered by finitely many balls of radius } r\},$$

where B denotes the unit ball in $W^{v,-h}_\omega$ with respect to the $\|\cdot\|_{v,-h,\omega}$ norm.

The *index of compactness* of $\mathcal{F}_t(\omega)$ is the quantity

$$\kappa(\omega) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{F}_t(\omega)\|_{ic(X)}. \tag{5.2}$$

Observe that for a single operator L, $\|L\|_{ic(X)} = 0$ if and only if L is compact. Moreover, $\|L\|_{ic(X)}$ is sub-multiplicative, which implies that by the Kingman sub-additive ergodic theorem, $\kappa(\omega)$ is identically a constant for ν-a.e. ω.

We can now establish

Lemma 5.4. *There exists $v, h \in \mathbb{R}^+$ for which the index of compactness κ of the Forni cocycle can be made strictly smaller than λ_1.*

Proof. We have

$$\|\mathcal{F}_t(\omega)\|_{ic(X)} \le \|k_t(\omega)\|_{v,-h,\omega} \le Ce^{-\frac{1}{2}\min\{v,h\}t}.$$

In particular, this implies that

$$\kappa(\omega) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{F}_t(\omega)\|_{ic(X)} \le -\frac{1}{2}\min\{v,h\}.$$

Since λ_1 is constant a.e. (in fact, we will prove that $\lambda_1 = 0$), we can find $v, h \in \mathbb{R}^+$ satisfying the inequality $\kappa < \lambda_1$ a.e. □

We will also need to show that

Lemma 5.5. *$W^{v,-h}_\omega$ has a separable dual.*

Proof. First, it is clear that $L^2(M,\omega)$ is separable since (M,ω) admits a finite, faithful measure, namely $\mathrm{Re}\,\omega \wedge \mathrm{Im}\,\omega$, and as a Hilbert space, it is reflexive. This implies that $W^{v,-h}_\omega$ has a separable dual, as desired. $\square$

The main result of this section is

Theorem 5.6. *For ν-a.e. ω, there exists a constant r, with $1 \leq r \leq \infty$, and* exceptional *Lyapunov exponents*

$$\lambda_1 > \cdots > \lambda_r > \kappa$$

with multiplicities

$$m_1, \ldots, m_r \in \mathbb{N},$$

and closed subspaces $V_1(\omega)$, ..., $V_r(\omega)$, $Z_\infty(\omega)$, such that

(1) $\dim(V_i(\omega)) = m_i$
(2) $\mathcal{F}_t(\omega)(V_i(\omega)) = V_i(g_t\omega)$ *and* $\mathcal{F}_t(\omega)(Z_\infty(\omega)) \subset Z_\infty(g_t\omega)$
(3) $Dom(\mathcal{F}_t(\omega)) = Z_\infty(\omega) \oplus V_1(\omega) \oplus \cdots \oplus V_r(\omega)$
(4) for every $f \in V_i(\omega)\backslash\{0\}$, $\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{F}_t(\omega)f\|_{v,-h,\omega} = \lambda_i$,
(5) for every $f \in Z_\infty(\omega)$, $\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{F}_t(\omega)f\|_{v,-h,\omega} \leq \kappa$,

Moreover, the $\|\cdot\|_{v,-h,\omega}$ norms of the projections $\Pi_i(\omega) : \mathrm{Dom}(\mathcal{F}_t(\omega)) \to V_i(\omega)$ associated to the splitting of $Dom(\mathcal{F}_t(\omega))$ are tempered with respect to g_t: that is, for ν-almost every $\omega \in X$, we have

$$\lim_{n\to\pm\infty} \frac{1}{t} \|\Pi_i(g_t\omega)\|_{W^{v,-h}\to W^{v,-h}} = 0.$$

Proof. Follows from the MET where the conditions required have already been checked above. We should also state here that the METs are stated in discrete time, but as stated by A. Blumenthal and S. Punshon-Smith in **[BPS23]**, the extension to continuous-time is standard and classical. $\square$

Finally, our aim is to show that the conclusion of Theorem 5.6 only depends on the value κ. That is, it is independent of the choice of the order function $m(x,\xi)$ (which in turn determines $v, -h$ and the radius of the cones around $\mathbb{R}\mathrm{Re}\,\omega$ and $\mathbb{R}\mathrm{Im}\,\omega$), so that all the *exceptional* Lyapunov exponents λ_i that are greater than κ, together with the corresponding subbundles $V_i(\omega)$, are *intrinsic*.

It will be convenient to substitute the notation $W^{v,-h}_\omega$ with $W^{m(x,\xi)}_\omega$ for the rest of this section. Moreover, the Forni cocycle also depends on the order function, and so we also write $\mathcal{F}^m_t(\omega)$ to highlight this difference.

First, let $\mathcal{O}_{v,-h}$ be the set of all order functions satisfying the properties listed in Lemma 4.1. Observe that such a set is non-empty. Let $m, m' \in \mathcal{O}_{v,-h}$, and assume that $m' \geq m$. Then we have that $W^{m'(x,\xi)}_\omega \subset W^{m(x,\xi)}_\omega$, and that $W^{m'(x,\xi)}_\omega$ is dense in

$W_\omega^{m(x,\xi)}$. Similarly, we have that $W_{g_t\omega}^{m'(g_t(x,\xi))} \subset W_{g_t\omega}^{m(g_t(x,\xi))}$, and that $W_\omega^{m'(g_t(x,\xi))}$ is dense in $W_\omega^{m(g_t(x,\xi))}$. This implies the following three inequalities:

$$\begin{aligned}
&\|\cdot\|_{m',\omega} \geq \|\cdot\|_{m,\omega},\\
&\|\cdot\|_{m',g_t\omega} \geq \|\cdot\|_{m,g_t\omega},\\
&\|\cdot\|_{\mathrm{Dom}(\rho_t^{m'}(\omega))} \geq \|\cdot\|_{\mathrm{Dom}(\rho_t^{m}(\omega))}.
\end{aligned}$$

To show that the Osceldets decomposition of $\mathrm{Dom}(\mathcal{F}_t^m(\omega))$ and $\mathrm{Dom}(\mathcal{F}_t^{m'}(\omega))$ coincide, we appeal to the following proposition due to Gonzalez-Tokman-Quas [**GTQ21**, Theorem A.1].

Proposition 5.7. *Let Dom*$(\rho_t^m(\omega)) = Z_\infty(\omega) \oplus \bigoplus_{i=1}^r V_i(\omega)$ *and Dom*$(\rho_t^{m'}(\omega)) = Z'_\infty(\omega) \oplus \bigoplus_{i=1}^r V'_i(\omega)$ *be the splittings associated to Dom*$(\rho_t^m(\omega))$ *and Dom*$(\rho_t^{m'}(\omega))$*, respectively, and let* $\{\lambda_i\}_{i=1}^r$ *and* $\{\lambda'_i\}_{i=1}^r$ *be the correspodning exceptional Lyapunov exponents. Then, if* $\max(\lambda_i, \lambda'_i) > \alpha := \max(\kappa, \kappa')$*, then*

- $\lambda'_i = \lambda'_i$*, and*
- *for* ν*-a.e.* ω*,* $V_i(\omega) = V'_i(\omega)$

Together with g_t-invariance of the $V_i(\omega)$'s, together with the flexibility of the choice of the order functions (as the radius of the cone around $\mathbb{R}\mathrm{Re}\,\omega$ can be taken to be arbitrarily small), we can conclude now

Corollary 5.8. *For* ν*-a.e.* ω*, any* $1 \leq i \leq r$*, and any* $f \in V_i(\omega)$*, we have that* $WF(f) \subseteq \mathbb{R}Re\,\omega$*.*

Corollary 5.9. *For* ν*-a.e.* ω*, any* $1 \leq i \leq r$*, and any* $f \in V_i^*(\omega)$*, we have that* $WF(f) \subseteq \mathbb{R}Im\,\omega$*.*

5.1. **Oseledets Expansion of the Distributional Pairing.** Let us make explicit the expansion of the dynamical distributional pairing arising from the Forni cocycle $\mathcal{F}_t(\omega)$ and its dual $\check{\mathcal{F}}_{-t}(\omega)$, acting on the anisotropic Sobolev bundle and its dual, respectively. We recall the pairing between $f \in W_\omega^{v,-h}$ and $g \in W_\omega^{-v,h}$, defined as

$$\langle f, g\rangle_{W_\omega^{v,-h}\times W_\omega^{-v,h}} := \langle \mathrm{Op}(a_m(x,\xi))f, \mathrm{Op}(a_m(x,\xi))^{-1}g\rangle_{L^2(M,\omega)}.$$

Let $\{V_i(\omega)\}_{1\leq i\leq r}$ denote the Oseledets decomposition of the anisotropic Sobolev bundle $W_\kappa^{v,-h}$, and $\{V_j^d(\omega)\}_{1\leq j\leq r}$ the corresponding dual decomposition of the dual bundle $W_\kappa^{-v,h}$, arising from the dual cocycle $\check{\mathcal{F}}_t(\omega)$. The cocycles act as follows:

$$\mathcal{F}_t(\omega) : V_i(\omega) \to V_i(g_t\omega), \quad \check{\mathcal{F}}_{-t}(\omega) : V_j^d(\omega) \to V_j^d(g_{-t}\omega).$$

We can write

$$W_\omega^{v,-h} \otimes W_\omega^{-v,h} = \bigoplus_{i,j=1}^{k} \left(V_i \otimes V_j^*\right) \oplus \left(Z_\infty \otimes V_j^*\right) \oplus \left(V_i \otimes Z_\infty^*\right) \oplus \left(Z_\infty \otimes Z_\infty^*\right) \quad (5.1)$$

Let $\Pi_i(\omega)$ denote the projection onto the Oseledets space $V_i(\omega)$, and similarly let $\Pi_j^d(\omega)$ denote the projection onto the dual space $V_j^d(\omega)$. We can obtain the following asymptotic expansion:

$$\langle \mathcal{F}_t(\omega) f, \check{\mathcal{F}}_{-t}(\omega) g\rangle_{W^{v,-h}_\omega \times W^{-v,h}_\omega} = \sum_{i,j=1}^{r} e^{(\lambda_i+\lambda_j^d)t+o(t)} \langle \Pi_i(\omega) f, \Pi_j^d(\omega) g\rangle_{W^{v,-h}_\omega \times W^{-v,h}_\omega} + \mathcal{R}_t(f,g),$$

where λ_i and λ_j^d are the Lyapunov exponents corresponding to the spaces $V_i(\omega)$ and $V_j^d(\omega)$, respectively. The remainder term $\mathcal{R}_t(f,g)$ arises from the components of the cocycle acting on the subspaces not associated with the exceptional Lyapunov spectrum. These include all tensor products involving at least one of $Z_\infty(\omega)$ or $\check{Z}_\infty(\omega)$.

6. Simplicity of the Zero Exponent

The purpose of this section is to prove that the 0 Lyapunov exponent is simple. That is, the dimension of the corresponding Osceledets subbundle is equal to 1. We begin with the following

Lemma 6.1. *0 is a Lyapunov exponent and all functions in $Dom(\hat{\mathcal{F}}_t) \cap L^2(M,\omega)$ belong to the corresponding Osceledets subbundle.*

Proof. Recall that, since $\mathrm{Op}(a_m \circ g_t)$ can be chosen to be self-adjoint and invertible up to sub-leading terms by [FRS08, Corollary 4], we have that

$$\|\mathcal{F}_t(\omega) f\|_{v,-h,g_t\omega} = \|f\|_{L^2(M,\omega)}$$

for all $f \in \mathrm{Dom}(\hat{\mathcal{F}}_t(\omega)) \cap L^2(M,\omega)$. We note the the constants cannot belong to V_∞ (corresponding to the essential spectrum of the Forni cocycle) since distributions in $\mathrm{Dom}(\hat{\mathcal{F}}_t(\omega))$ belonging to V_∞ can only be exponentially contracting w.r.t the $\|\cdot\|_{v,-h,g_t}$ norm by an exponential rate that is at most $\frac{-\min\{v,h\}}{2}$.

Since the constants belong to $L^2(M,\omega)$, we see that 0 is then a Lyapunov exponent of our Forni cocycle and the corresponding Osceledets bundle is at least 1-dimensional. □

One also needs to rule out sublinear growth:

Lemma 6.2. *Let $f \in Dom(\hat{\mathcal{F}}_t) \cap L^2(M,\omega)$. Then $\log \|\mathcal{F}_t(\omega) f\|_{L^2}$ cannot grow sublinearly.*

Proof. Suppose, for the sake of contradiction, that $\log \|\mathcal{F}_t(\omega) f\|_{L^2} = o(t)$, so that $\frac{1}{t}\log \|\mathcal{F}_t(\omega) f\|_{L^2} \to 0$ as $t \to \infty$. This is clearly impossible since $\|\mathcal{F}_t(\omega) f\|_{L^2} = \|f\|_{L^2}$ and $\|f\|_{L^2} < \infty$. □

The main result of this section is

Proposition 6.3. . *Let $f \in Dom(\hat{\mathcal{F}}_t)$. If $\mathcal{F}_t(\omega) f = \lambda(t) f$, with $|\lambda(t)| = 1$, then $f \in C^\infty(M)$. In other terms, we will show that for $f \in Dom(\hat{\mathcal{F}}_t)$, if*

$$\mathrm{Op}(a_m \circ g_t) f = \lambda(t) Op(a_m) f,$$

with $|\lambda(t)| = 1$, then $f \in C^\infty(M)$.

Before proving Proposition 6.3, let us show the following corollary, based on [FRS08, Lemma 3]:

Corollary 6.4. *If f satisfies the assumptions of Theorem 6.3, then f is identically a constant, and therefore the zero Lyapunov exponent is simple.*

Proof. First, by compactness of M, we have $|\mathcal{F}_t(\omega)f|_{L^\infty} = |f|_{L^\infty} < \infty$. We remark that the action of the cocycle extends to vector fields and Lie derivatives: it scales the vector fields $X = (dx)^*$ and $Y = (dy)^*$ by e^{-t} or e^t, respectively. Then, we observe that $|\mathcal{F}_t(\omega)L_Y f| = |L_{e^t Y} f| = e^t |L_Y f|$ for all $t \in \mathbb{R}$. Since, for $f \in C^1(M)$, we have $|\mathcal{F}_t(\omega)L_Y f| = |L_Y f| < \infty$, this implies $e^t |L_Y f dx| \to \infty$ as $t \to \infty$ unless $|L_Y f| = 0$, which in turn implies that $L_Y f = 0$. Similarly, we observe that $|\mathcal{F}_t(\omega)L_X f| = |L_{e^{-t}X} f| = e^{-t}|L_X|$ for all $t \in \mathbb{R}$. Since, for $f \in C^1(M)$, we have $|\mathcal{F}_t(\omega)L_X f| = |L_X f| < \infty$, this implies $e^{-t}|L_X f| \to \infty$ as $t \to -\infty$ unless $|L_X f| = 0$, which in turn implies that $L_X f = 0$. Since M is connected, this implies that f is identically a constant. □

We are now ready to prove Theorem 6.3, based on an adapation of [FRS08, Lemma 4] to our setting:

Proof of Proposition 6.3. We will use an h-semiclassical quantization given by

$$\mathrm{Op}_h(a(x,\xi))u(x) = \frac{1}{(2\pi h)^2} \int e^{i\frac{(x-y)\xi}{h}} a(x,\xi)u(y)dxd\xi \tag{6.1}$$

First, there exist a partition of unity in S^0 with symbols

$$0 \le B(x,\xi), C(x,\xi) \in S^0,$$

such that

$$1 = B^2 + C^2, \tag{6.2}$$

and $B(x,\xi) = 1$ on the set

$$R_{2t}^2 \le |\xi_x|^2 \le R_{2t}^2(1-\delta)(1+|\xi_y|^2) \tag{6.3}$$

with support in

$$R_{2t}^2 \le |\xi_x|^2 \le R_{2t}^2(1+\delta)(1+|\xi_y|^2), \tag{6.4}$$

where $\delta > 0$ is small enough.

Moreover, $(B \circ g_t)$ is equal to 1 on a set

$$R_t^2 \le |\xi_x|^2 \le a(t)R_t^2(1+|\xi_y|^2) \tag{6.5}$$

and has its support in a set

$$R_t^2 \le |\xi_x|^2 \le b(t)R_t^2(1+|\xi_y|^2), \tag{6.6}$$

where $1 < a(t) < b(t)$ are independent of δ for $\delta > 0$ small enough.

It follows that we can write

$$a_m^2 = (Ba_m)^2 + (Ca_m)^2, \tag{6.7}$$

and $B(x,\xi)a_m = a_m$ on the set

$$|\xi_x|^2 \le (1-\delta)(1+|\xi_y|) \tag{6.8}$$

with support in

$$|\xi_x|^2 \leq (1+\delta)(1+|\xi_y^2|), \tag{6.9}$$

where $\delta > 0$ is small enough.

Moreover, $(Ba_m) \circ g_t$ is equal to $a_m \circ g_t$ on a set

$$|\xi_x|^2 \leq a(t)(1+|\xi_y|^2) \tag{6.10}$$

and has its support in a set

$$|\xi_x|^2 \leq b(t)(1+|\xi_y|^2), \tag{6.11}$$

where $1 < a(t) < b(t)$ are independent of δ for $\delta > 0$ small enough.

Now we can construct corresponding h-pseudodifferential operators such that

$$\mathrm{Op}_h(a_m)^2 = \mathrm{Op}_h(Ba_m)^2 + \mathrm{Op}_h(Ca_m)^2 + K, \tag{6.12}$$

where K is negligible in the sense that

$$K = O(h^N) : H^{-N} \to H^N, \forall N \in \mathbb{N}, \tag{6.13}$$

and such that the symbol of $\mathrm{Op}_h(Ba_m)$ is equal to Ba_m modulo hS^{m-1}, and modulo $h^\infty S^{-\infty}$ it is equal to a_m on the set 6.8 and has its support in the set 6.9.

It follows that

$$\mathrm{Op}_h(a_m \circ g_t)^2 = \mathrm{Op}_h(Ba_m \circ g_t)^2 + \mathrm{Op}_h(Ca_m \circ g_t)^2 + K,$$

and such that the symbol of $\mathrm{Op}_h(Ba_m) \circ g_t$ is equal to Ba_m modulo hS^{m-1} and modulo $h^\infty S^{-\infty}$ it is equal to a_m on the set 6.10 and has its support in the set 6.11.

Taking the difference, we can find a self-adjoint PDO $\hat{D}$ such that:

$$\mathrm{Op}_h(a_m \circ g_t)^2 - \mathrm{Op}_h(a_m)^2 = \mathrm{Op}_h(Ba_m \circ g_t)^2 - \mathrm{Op}_h(Ba_m)^2 + \hat{D}^2 + L \tag{6.14}$$

In fact, we choose the symbol D of $\hat{D} = \mathrm{Op}_h(D(x,\xi))$ so that $\mathrm{Op}_h(D(x,\xi))^2$ is equal to zero in the region where $B = 1$, and such that $\mathrm{Op}_h(D(x,\xi))^2$ is equal to

$$(1 - [\mathrm{Op}_h(a_m)^2 - \mathrm{Op}_h(Ba_m)^2] + [\mathrm{Op}_h(a_m \circ g_t)^2 - \mathrm{Op}_h((Ba_m) \circ g_t)^2])$$

in the support of B where B is not identically equal to 1, and finally, $\mathrm{Op}_h(D(x,\xi))^2$ equals

$$\mathrm{Op}_h(a_m \circ g_t) - \mathrm{Op}_h((Ba_m) \circ g_t)$$

outside the support of B. In fact, by construction, we have that $\hat{D}$ is semi-classically elliptic in the region

$$(1+\delta)(1+|\xi_y|^2) \leq |\xi_x|^2 \leq a(t)(1+|\xi_y|^2).$$

for $|\xi_x| > R_t$.

Now, let $f \in \mathrm{Dom}(\mathcal{F}_t(\omega))$ be such that

$$\mathrm{Op}_h(a_m \circ g_t) f = \lambda(t) \, \mathrm{Op}_h(a_m) f.$$

Observe that since $\mathrm{Op}_h((Ba_m) \circ g_t) f, \mathrm{Op}_h(Ba_m) f$ also belong to $L^2(M,\omega)$, it follows by g_t-invariance that

$$\| \mathrm{Op}_h((Ba_m) \circ g_t) f\|_{L^2} = \| \mathrm{Op}_h(Ba_m)) f\|_{L^2}.$$

Observe that

$$\langle (\mathrm{Op}_h(a_m \circ g_t)^2 - \mathrm{Op}_h(a_m)^2)f, f\rangle = \|\mathrm{Op}_h(a_m \circ g_t)f\|^2 - \|\mathrm{Op}_h(a_m)f\|^2 = 0. \quad (6.15)$$

Similarly, we have

$$\langle (\mathrm{Op}_h((Ba_m) \circ g_t)^2 - \mathrm{Op}_h(Ba_m)^2)f, f\rangle = \|\mathrm{Op}_h((Ba_m) \circ g_t)f\|^2 - \|\mathrm{Op}_h(Ba_m)f\|^2 = 0 \quad (6.16)$$

implying therefore that 6.14 reduces to

$$\|\hat{D}f\|^2 = -\langle Lf, f\rangle.$$

Since L is negligible, we have

$$\|\hat{D}f\|^2 = O(h^\infty). \quad (6.17)$$

Since $\hat{D}$ is semi-classically elliptic in the region

$$(1+\delta)(1+|\xi_y|^2) \leq |\xi_x|^2 \leq a(t)(1+|\xi_s|^2),$$

we see that, by microlocal elliptic regularity (express this using wavefront sets), f is microlocally $O(h^\infty)$ in the region (replacing $\xi \to h\xi$)

$$(1+\delta)(\frac{1}{h^2}+|\xi_y|^2) \leq |\xi_x|^2 \leq a(t)(\frac{1}{h^2}+|\xi_y|^2),$$

and letting $h \to 0$, we see that f has no wave-front set in a conical neighborhood of dx (for f that is smooth everywhere except along the horizontal direction, which is a property satisfied by *all* distributions belonging to Osceledets subbundles, see Lemma (to be written)). Since we already know that $\mathrm{WF}(f) \subset \mathbb{R}\mathrm{Re}\omega$, we conclude that $f \in C^\infty$. ☐

One still needs to ensure out that

Proposition 6.5. 0 *is the largest Lyapunov exponent.*

Proof. Suppose $f, g \in C^\infty(M)$. It follows by the compactness of M that

$$\langle \mathcal{F}_t(\omega)f, \check{\mathcal{F}}_{-t}(\omega)g\rangle_{W^{v,-h}\otimes W^{-v,h}} = \langle f, g\rangle_{L^2} \leq \|f\|_\infty \|g\|_\infty < \infty. \quad (6.18)$$

This implies that that

$$\frac{1}{t}\log\langle \mathcal{F}_t(\omega)f, \check{\mathcal{F}}_{-t}(\omega)g\rangle_{W^{v,-h}\otimes W^{-v,h}} \leq 0$$

as $t \to \infty$. Since $C^\infty(M)$ is dense in $W^{v,-h}(M)$ and $W^{-v,h}(M)$, and the pairing

$$(f,g) \in W^{v,-h} \otimes W^{-v,h} \mapsto \langle \mathcal{F}_t(\omega)f, \check{\mathcal{F}}_{-t}(\omega)g\rangle_{W^{v,-h}\otimes W^{-v,h}}$$

is continuous, this implies that the same bound is true for all $(f,g) \in W^{v,-h} \otimes W^{-v,h}$, and thus 0 is the largest Lyapunov exponent.

☐

7. A Sobolev Trace Theorem

In the sequel, we follow closely the exposition in Forni (see, for e.g., Lemma 2.3 in [?]). For any Abelian differential $\omega \in \mathcal{H}(\kappa)$, let $\delta(\omega)$ denote the length of the shortest saddle connection on the translation surface (M, ω), and let R_ω be the flat metric induced by the Abelian differential ω. We will now prove

Lemma 7.1 (Sobolev trace theorem)**.** *For any stratum $\mathcal{H}(\kappa)$ and any $h, v > 1/2$ there exists a constant $C_{\kappa,h,v} > 0$ such that the following holds. For any Abelian differential $\omega \in \mathcal{H}(\kappa)$, any horizontal or vertical regular geodesic segment $\gamma \subset M$ of finite R_ω-length $L_\omega(\gamma)$ defines by integration a current of degree 1 (and dimension 1) $\gamma \in \Omega^1 W_\omega^{-v,h}(M)$ such that*

$$|\gamma|_{\Omega^1 W_\omega^{-v,h}(M)} \leq C_{\kappa,h,v}\left(1 + \frac{L_\omega(\gamma)}{\delta(\omega)}\right).$$

Proof. We adapt Forni's proof closely, with some modifications that take into account our anisotropic Sobolev trace theorem for rectangles, proven in the appendix. Exactly as in Forni's proof, and up to notational modifications, the regular (horizontal) arc γ can be decomposed as a union $\gamma = \cup_{i=0}^{N} \gamma_i$ of consecutive (horizontal) sub-arcs such that the following properties hold:

(1) the length $L_\omega(\gamma_i)$ of the arcs γ_i, with respect to the flat metric R_ω induced by the Abelian differential ω, satisfy the bounds
$$\delta(\omega)/3 \leq L_\omega(\gamma_i) \leq 2\delta(\omega)/3\,, \text{ for all } i \in \{1, \dots, N-1\}\,,$$
$$L_\omega(\gamma_0)\,, L_\omega(\gamma_N) \leq 2\delta(\omega)/3;$$

(2) the rectangle $R_i = [0, L_\omega(\gamma_i)] \times (-\delta(\omega)/3, \delta(\omega)/3) \subset \mathbb{R}^2$ embeds isometrically in the flat surface $(M \setminus \Sigma_\omega, R_\omega)$, so that the arc $\bar{\gamma}_i := [0, L_\omega(\gamma_i)] \times \{0\}$ has image equal to $\gamma_i \subset M$, for all $i \in \{0, \dots, N\}$.

Proceeding as in Forni, we now derive the statement from the anisotropic Sobolev trace theorem (in $\mathbb{R}^2$) applied to each arc $\bar{\gamma}_i \subset R_i \subset \mathbb{R}^2$, for $i \in \{0, \dots, N\}$. Let

$$R_{a,b} := \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < a\,,\ -b < y < b\}\,.$$

We first analyze the case $R_{1,1}$. Note that Theorems A.2 and A.5 ensure that there exists a continuous map

$$\tau_0 : W^{v,-h}(R_{1,1}) \to W^{-\sigma}((0,1)),$$

for some index $\sigma > h + 1$ satisfying

$$\|\tau_0 f\|_{W^\sigma((0,1))} \leq C\|f\|_{W^{v,-h}(R_{1,1})},$$

for all $f \in W^{v,-h}(R_{1,1})$ and some uniform constant $C > 0$. The existence of this continuous operator implies that its adjoint operator $\tau_0^* : W^\sigma((0,1)) \to W^{v,-h}(R_{1,1})$ is also bounded, where the reference measure is the Lebesgue measure on the translation surface. We will now bound the integral

$$\int_0^1 f(x, 0)dx.$$

for $f \in W^{v,-h}_\omega(R_{1,1})$. This integral can be rewritten as follows:

$$\langle 1, \tau_0 f\rangle_{W^\sigma \otimes W^{-\sigma}} = \int_0^1 f(x,0)\, dx = \langle \tau_0^*(1), f\rangle_{W^{v,-h}\otimes W^{-v,h}}$$

where the last equality follows by duality. We now derive the bound

$$\left|\int_0^1 f(x,0)\, dx\right| = |\langle \tau_0^*(1), f\rangle_{W^{v,h}\otimes W^{v,h}}| \le \|\tau_0^*(1)\|_{W^{-v,h}_\omega(R_{1,1})} \|f\|_{W^{v,-h}_\omega(R_{1,1})}$$

with

$$\|\tau_0^*(1)\|_{W^{-v,h}_\omega(R_{1,1})} \le C\|1\|_{W^\sigma_\omega((0,1))} = C$$

and hence we can write

$$\left|\int_0^1 f(x,0)\, dx\right| \le C\|f\|_{W^{v,-h}_\omega(R_{1,1})}.$$

Exactly as in Forni, we can now derive the following bound: for every $h, v > 1/2$ there exists a constant $K_{h,v} > 0$,

$$\Big| \int_0^a f(x,0)dx \Big| \le K_{h,v} \left(\frac{a}{b}\right)^{1/2} \max\{a,b,1\}^s \, \|f\|_{W^{v,-h}_\omega(R_{a,b})} \,,$$

for any $s > \max\{h, v\}$.

Hence, by taking into account that the systole function is uniformly bounded above on each stratum [**?**, Corollary 5.6], we conclude that there exists a constant $C_{\kappa,h,v} > 0$ such that

$$|\gamma_i|_{\Omega^1 W^{-v,h}_\omega(M)} := \sup_{\substack{f\in W^{v,-h}_\omega \\ \|f\|_{W^{v,-h}_\omega}\le 1}} |\langle \gamma_i, f dx\rangle| \le C_{\kappa,h,v}\,, \qquad \text{for all } i \in \{0,\dots,N\}\,.$$

The estimate in the statement then follows by taking into account the inequality $N-1 \le 3L_\omega(\gamma)/\delta(\omega)$, which is an immediate consequence of the above lower bounds on the lengths of the sub-arcs γ_i for $i \in \{1,\dots,N-1\}$. □

8. Quantitative Unique Ergodicity

The purpose of this section is to prove the main theorem of our paper, which we recall here:

Theorem 8.1. *Assume that the forward Teichmüller orbit $g_{\mathbb{R}^+}(M,\omega)$ visits a compact set $K \subset \mathcal{H}_\kappa$ with positive frequency, that is,*

$$f_K := \liminf_{t\to+\infty} \ \mathrm{Leb}(\{t \ge 0 | g_{-t}(M,\omega) \in K\}) > 0\,.$$

Then there exist constants $C(M,\omega) > 0$ and $\alpha > 0$ such that, for h, v sufficiently large, for all distributions $f \in W^{-v,h}(M)$ of zero average and for all $(p,T) \in M\times\mathbb{R}^+$, such that p has an infinite forward orbit under $\phi^X_{\mathbb{R}}$, we have

$$\Big|\frac{1}{T}\int_0^T f\circ\phi^X_t(p)dt\Big| \le C(M,\omega)\|f\|_{\mathcal{W}^{v,-h}(M)}\, T^{-\alpha}\,.$$

First, some preliminaries. Let Ω^1_ω be the vector space of *tempered* differential 1-forms on (M,ω), and let $D'_\omega(M)$ be its dual. Let

$$\Omega^1 W^{v,-h}_\omega(M) := W^{v,-h}_\omega(M) \otimes \Omega^1_\omega(M) \subseteq D'_\omega(M)$$

denote the space of *tempered* anisotropic currents on M with distributional coefficients in $W^{v,-h}_\omega$. We endow these anisotropic 1-currents in $\Omega^1 W^{v,-h}_\omega(M)$ with the following anisotropic norm: for every anisotropic 1-form $\alpha \in W^{v,-h}_\omega(M) \otimes \Omega^1_\omega(M)$,

$$|\alpha|_{\Omega^1 W^{v,-h}_\omega(M)} := \left(\|\imath_X \alpha\|^2_{W^{v,-h}_\omega(M)} + \|\imath_Y \alpha\|^2_{W^{v,-h}_\omega(M)} \right)^{1/2} .$$

Analogous to the construction of the bundle of anisotropic distributions over a connected component of the stratum $\mathcal{H}_\kappa$, the above considerations give rise to the (sub)bundles Ω^1_κ, $D'_\kappa(M)$, and $\Omega^1 W^{v,-h}_\kappa(M)$, together with a Forni cocycle acting on these (sub)bundles of anisotropic 1-currents.

As a consequence of the spectral gap of the Forni coycle acting on on the subbundle $W^{v,-h}_\kappa(M)$ of anisotropic distributions, i.e. the simplicity of the top Lyapunov exponent, we can also derive a spectral gap result for its action on the subbundle $\Omega^1 W^{v,-h}_\kappa(M)$ of anisotropic 1-currents. We have

Lemma 8.2. *The adjoint $(\check{\mathcal{F}}_{\mathbb{R}^-}(\omega))^*$ of the dual Forni cocycle $\check{\mathcal{F}}_{\mathbb{R}^-}(\omega)$ acting on $\Omega^1 W^{v,-h}_\kappa(M)$ has a spectral gap, in the following sense: if the forward Teichmüller orbit $g_{\mathbb{R}^+}(M,\omega)$ visits a compact set $K \subset \mathcal{H}_\kappa$ with positive frequency, in the sense that*

$$f_K := \liminf_{t\to+\infty} \ \text{Leb}(\{t \geq 0 | g_t(M,\omega) \in K\}) > 0 \,, \tag{8.1}$$

then, there exist constants $C > 0$ and $\alpha > 0$ such that, for all $\gamma \in \Omega^1 W^{v,-h}_\kappa(M)$, such that $\langle \gamma, Re(\omega)\rangle = \langle \gamma, Im(\omega)\rangle = 0$, and for all $t > 0$,

$$|(\check{\mathcal{F}}_{-t}(\omega))^* \mathcal{F}_t(\omega)(\gamma)|_{\Omega_1 W^{v,-h}_\omega(M)} \leq C e^{(1-\alpha)t} |\mathcal{F}_t(\omega)(\gamma)|_{\Omega_1 W^{v,-h}_\omega(M)} \,.$$

.

Proof of Theorem 8.1. We write ergodic integrals of the horizontal translation flow in terms of the anisotropic 1-currents $\gamma^Y_T(p) \in \Omega^1 W^{v,-h}_\omega(M)$, arising from the ergodic integral

$$\int_0^T f \circ \phi^X_t(p) dt = \langle \gamma^X_T(p), f \text{Re}(\omega)\rangle. \tag{8.2}$$

We consider a sequence (t_n) of return times of the forward Teichmüller orbit $g_{\mathbb{R}^+}$ to K with positive frequency, that is, such that

$$\lim_{n\to+\infty} \frac{t_n}{n} = f_K > 0 \,. \tag{8.3}$$

Since $g_{t_n}(M,h) \in K$, there exists a constant $C_K > 0$, such that

$$\frac{1}{C_K} \leq T_n(p) e^{-t_n} \leq C_K,$$

and we study the ergodic integral at times T_n:

$$\int_0^{T_n} f \circ \phi_t^X(p)dt = \langle \gamma_{T_n}^X(p), f\mathrm{Re}(\omega)\rangle. \tag{8.4}$$

We renormalize the trajectory using the action of the Teichmüller flow g_{t_n}, which expands $\mathrm{Re}\,\omega$ by e^{t_n}, so that the horizontal vector field transforms as

$$X_{g_{t_n}\omega} = e^{-t_n} X_\omega.$$

Consequently, we have the change-of-variable identity:

$$\Omega_1 W_\omega^{v,-h} \ni \gamma_{T_n}^X(p) = \gamma_{e^{-t_n}T_n}^{X_{g_{t_n}\omega}}(p) \in \Omega_1 W_{g_{t_n}\omega}^{v,-h},$$

where the right-hand side is the current obtained by flowing along the rescaled vector field $X_{g_{t_n}\omega}$ for time $e^{-t_n}T_n$ on the surface $(M_{t_n}, g_{t_n}\omega)$.
We also note that, by definition of the extension of the Forni cocycle to the space of anisotropic 1-currents, we have

$$|\gamma_{e^{-t_n}T_n}^{X_{g_{t_n}\omega}}(p)|_{\Omega^1 W_{g_t\omega}^{v,-h}(M)} = |(\check{\mathcal{F}}_{-t_n}(\omega))^* \mathcal{F}_{t_n}(\omega)(\gamma_T^X(p))|_{\Omega^1 W_\omega^{v,-h}},$$

so we can derive, by Lemma 8.1,

$$|(\check{\mathcal{F}}_{-t_n}(\omega))^* \mathcal{F}_{t_n}(\omega)(\gamma_{T_n}^X(p))|_{\Omega^1 W_\omega^{v,-h}} \leq C_1 e^{(1-\alpha)t_n} |\mathcal{F}_{t_n}(\omega)(\gamma_{T_n}^X(p))|_{\Omega^1 W_\omega^{v,-h}(M)}.$$

On the other hand, by the anisotropic Sobolev trace theorem, we have

$$|\mathcal{F}_{t_n}(\omega)(\gamma_{T_n}^X(p))|_{\Omega^1 W_\omega^{v,-h}(M)} \leq C_2 \left(1 + \frac{e^{-t}T_n}{\delta(g_{t_n}\omega)}\right),$$

where $\delta(g_{t_n}\omega)$ denotes the length of the shortest saddle connection on $(M_{t_n}, g_{t_n}\omega)$.

Combining these estimates, we obtain

$$|\mathcal{F}_{t_n}(\omega)(\gamma_{e^{-t_n}T_n}^X(p))|_{\Omega^1 W_\omega^{-v,h}} \leq C_1 C_2 e^{(1-\alpha)t_n} \left(1 + \frac{e^{-t}T_n}{\delta(g_{t_n}\omega)}\right).$$

Since the length of the shortest saddle connection $\delta(g_{t_n}\omega)$ is uniformly bounded from below by a constant that depends on our compact set K, there exists a constant $C_4 > 0$ such that

$$\left|\int_0^{T_n} f \circ \phi_s^X(p)ds\right| \leq C_4 T_n^{1-\alpha} \|f\|_{W_\omega^{v,-h}} \tag{8.5}$$

By a standard decomposition argument (a generalization of the so-called Ostrowski expansion of an irrational number, see [For02]), by the condition in formula (**??**), it is then possible to extend the estimate (8.5) on ergodic integrals to arbitrary times. □

8.1. Smooth Solutions of the Cohomological Equation.

Remark 8.3. As with the horizontal flow, this will necessitate that the vertical flow be uniquely ergodic, and that the backward and forward vertical flow be defined for all T. Indeed, the set of ω whose horizontal and vertical foliations are uniquely ergodic is of full measure with respect to any ergodic g_t-invariant probability measure in the stratum. These considerations will be revisited in Section 9.

Appendix A. An Adapted Order Function

In this appendix, we give a proof of Lemma 4.14, which introduces our order function. We recall its content for the convenience of the reader.

Lemma A.1. *Let $v \in \mathbb{R}^+$ and $h \in \mathbb{R}^+$. There exists an order function $m_\omega \in S^0_1 \subset C^\infty_\omega(T^*M)$, with values in the interval $[-h, v]$, that fulfills the following properties. For all $\xi \in \mathbb{R}^2 \setminus \{0\}$, with $|\xi| \geq 1$, m is defined projectively, i.e., it only depends on the direction $\xi/|\xi|$. Moreover, for all $\xi \in \mathbb{R}^2$, $m(\xi) \equiv v$ in a conical neighborhood of $Re(\omega)$, and $m(\xi) \equiv -h$ in a conical neighborhood of $Im(\omega)$, for $|\xi| > 1$. In addition,*

$$\forall t > 0, \; m_{g_t\omega}(\xi) - m_\omega(\xi) \leq 0. \tag{A.1}$$

Let us define

$$a^\omega_m(\xi) := \langle \xi \rangle^{m_\omega(\xi)},$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. a^ω_m is a symbol of class $S^{m_\omega(\xi)}_\rho$ and it satisfies the following: for all $t > 0$, there exists $R_t \in \mathbb{R}^+$, depending on t, such that

$$\frac{a^{g_t\omega}_m(\xi)}{a^\omega_m(\xi)} < Ce^{-\frac{1}{2}\min\{v,h\}t} \tag{A.2}$$

for $|\xi| > R_t$ and a uniform constant $C > 0$.

Proof. Let $z \in \mathbb{C}$ and let $\{dx, dy\}$ be the basis of the cotangent bundle $T^*_z\mathbb{C}$. We construct a function $m(z, \xi)$ which is independent of the point $z \in \mathbb{C}$, but only depends on $\xi \in T^*_z\mathbb{C}$. Since we require $m(z, \xi) = m(\xi)$ to be defined projectively for $|\xi| \geq 1$, we can just assign values to m on $S^1 = \{\xi \in T^*_z\mathbb{C} \equiv \mathbb{C} : \; |\xi| = 1\}$, and then define $m(z, \xi) = m(z, \xi/|\xi|)$ for $|\xi| > 1$.

Let V_v, resp. V_h, be the restriction to S_1 of a conical neighborhood of dy, resp. dx, such that $V_v \cap V_h = \varnothing$. We firstly introduce the following function $m_o \in C^\infty(S^1, [-h, v])$ such that $m_o = v$ in V_v, $m_o = -h$ in V_h and m_o is decreasing along the arcs connecting V_v to V_h. Let $T > 0$ to be fixed later in the proof such that $W_v := S^1 \setminus g_{-T}(V_h) \subset V_v$ and $W_h := S^1 \setminus g_T(V_v) \subset V_h$. With a slight abuse of notation we use the same symbol to denote the g_t action on $T^*_z\mathbb{C} \equiv \mathbb{C}$ and the restriction to S^1. In the latter case, every covector is rescaled by his norm. We define the following function of S^1

$$m(\xi) := \frac{1}{2T}\int_{-T}^{T} m_o \circ g_s(\xi) ds.$$

For $\xi \in S^1$, we introduce the excursion time

$$\tau(\xi) := \max\{t \in \mathbb{R} : \; g_t\xi \in S^1 \setminus (V_v \cup V_h)\} - \min\{t \in \mathbb{R} : \; g_t\xi \in S^1 \setminus (V_v \cup V_h)\}$$

and let $\tau = \max\{\tau(\xi) : \; \xi \in S^1\} < \infty$. Notice that τ is independent of T. We have the following inequalities:

(1) If $\xi \in S^1 \setminus (W_v \cup W_h)$, then, denoting by G the generator of g_t

$$\begin{aligned} Gm(\xi) =& \frac{1}{2T}(m_o \circ g_{2T}(\xi) - m_o \circ g_{-2T}(\xi)) = \\ =& \frac{1}{2T}(-h-v) < -\frac{1}{2}\min\{v,h\} < 0, \end{aligned}$$

assuming T large enough.

(2) If $\xi \in W_v$ and $\tau < T$, then

$$\begin{aligned} m(\xi) =& \frac{1}{2T}\left(\int_{-T}^{T-\tau} \underbrace{m_o \circ g_s(\xi)}_{=v} ds + \int_{T-\tau}^{T} \underbrace{m_o \circ g_s(\xi)}_{\geq -h} ds\right) > \\ >& \frac{(2T-\tau)v}{2T} - \frac{\tau h}{2T} > \frac{v}{2}, \end{aligned}$$

for T large enough.

(3) If $\xi \in W_h$ and $\tau < T$, then

$$\begin{aligned} m(\xi) =& \frac{1}{2T}\left(\int_{-T}^{-T+\tau} \underbrace{m_o \circ g_s(\xi)}_{\leq v} ds + \int_{-T+\tau}^{T} \underbrace{m_o \circ g_s(\xi)}_{=-h} ds\right) < \\ <& \frac{\tau v}{2T} - \frac{(2T-\tau)h}{2T} < -\frac{h}{2}, \end{aligned}$$

for T large enough.

We remark that, by construction, there exist conical neighborhoods $\overline{W}_s \subset W_v \subset V_v$ and $\overline{W}_u \subset W_h \subset V_h$ such that $m|_{\overline{W}_s} = v$ and $m|_{\overline{W}_u} = -h$. Finally, we extend m to a smooth function on $T^*_z\mathbb{C}$ such that $m(\xi) = 0$ for $|\xi| \leq 1/2$ and $m(\xi) = m(\xi/|\xi|)$ for $|\xi| > 1$. By construction, $m|_{S_1}$ is nonincreasing along the orbits of g_t. Without loss of generality, we can assume the same for m on $T^*_z\mathbb{C}$. Notice that $m \in S^0_1 \subset C^\infty(T^*M)$ in the sense of Definition 3.3, i.e., it is a well-defined order function that satisfies (A.1).

Let us consider the escape function a_m. As above, we can treat a_m as function on $T^*\mathbb{C}$ and then lift it to a function on T^*M. As a consequence of [**FRS08**, Lemma 6], the function $a_m \in C^\infty(T^*\mathbb{C})$ is a symbol of variable order of class $S^{m(x,\xi)}_\rho$, for any $\rho \in [1/2, 1)$. We are left with the proof of (A.2).

Let us fix $R_t > 0$ such that $|g_s(\xi)| > 1$ for all $s \in [0,t]$, so that m is always defined radially. We must distinguish between covectors in $T^*_z\mathbb{C}$ and their rescaling to S^1. Thus, we write $\tilde{\xi} := \xi/|\xi|$ and $\tilde{g}_t\tilde{\xi} := g_t\xi/|g_t\xi|$. We limit ourselves to consider the case $\tilde{\xi} \in W_v$ and $\tilde{g}_t\tilde{\xi} \in W_h$. The other cases, $\tilde{\xi} \in S^1 \setminus (W_v \cup W_h)$ and $\tilde{\xi} \in W_h$, can be proved with a simpler version of the following argument. Let $t_1 := \max\{s \in [0,t] : \tilde{g}_t\tilde{\xi} \in W_v\}$ and $t_2 := \max\{s \in [t_1, t] : \tilde{g}_t\tilde{\xi} \in S^1 \setminus (W_v \cup W_h)\}$. We can write

$$\frac{a_m \circ g_t(x,\xi)}{a_m(x,\xi)} = \frac{a_m \circ g_{t_1}(x,\xi)}{a_m(x,\xi)} \cdot \frac{a_m \circ g_{t_2}(x,\xi)}{a_m \circ g_{t_1}(x,\xi)} \cdot \frac{a_m \circ g_t(x,\xi)}{a_m \circ g_{t_2}(x,\xi)}. \tag{A.3}$$

Let us study every factor of (A.3) independently. The first factor represents the flow of $\tilde{\xi}$ in W_v. Accordingly,

$$\frac{a_m \circ g_{t_1}(\xi)}{a_m(\xi)} = \frac{\langle g_{t_1}\xi\rangle^{m(g_{t_1}\xi)}}{\langle\xi\rangle^{m(\xi)}} \leq \left(\frac{\langle g_{t_1}\xi\rangle}{\langle\xi\rangle}\right)^{m(\xi)} \leq \left(\frac{\langle g_{t_1}\xi\rangle}{\langle\xi\rangle}\right)^{\frac{v}{2}} =$$
$$= \left(\frac{1+e^{2t_1}\xi_x^2+e^{-2t_1}\xi_y^2}{1+\xi_x^2+\xi_y^2}\right)^{\frac{v}{4}} \leq \left(\frac{1+2e^{-2t_1}\xi_y^2}{\xi_y^2}\right)^{\frac{v}{4}} \leq \left(\frac{4e^{-2t_1}\xi_y^2}{\xi_y^2}\right)^{\frac{v}{4}} \leq C_1 e^{-t_1\frac{v}{2}},$$

where we used $\langle g_{t_1}\xi\rangle \leq \langle\xi\rangle$, $m(g_{t_1}\xi) \leq m(\xi)$, $e^t\xi_x \leq e^{-t}\xi_y$ and $m(\xi) > \frac{v}{2}$ from the inequality (2) above.

To study the second factor of (A.3), we notice that the inequality (1) above gives

$$m \circ g_{t_2}(\xi) - m \circ g_{t_1}(\xi) < -\frac{t_2-t_1}{2}\min\{v,h\}.$$

Since the quotient $\langle g_{t_2}(\xi)\rangle / \langle g_{t_1}(\xi)\rangle$ is uniformly bounded in the cone connecting W_v to W_h, we obtain

$$\frac{a_m \circ g_{t_2}(x,\xi)}{a_m \circ g_{t_1}(x,\xi)} < C_2 e^{-\frac{t_2-t_1}{2}\min\{v,h\}}.$$

Lastly, $g_s(\xi) \in W_h$ for $s \in [t_2, t]$. Thus,

$$\frac{a_m \circ g_t(\xi)}{a_m \circ g_{t_2}(\xi)} = \frac{\langle g_t\xi\rangle^{m(g_t\xi)}}{\langle g_{t_2}\xi\rangle^{m(g_{t_2}\xi)}} \leq \left(\frac{\langle g_t\xi\rangle}{\langle g_{t_2}\xi\rangle}\right)^{m(g_t(\xi))} \leq \left(\frac{\langle g_t\xi\rangle}{\langle g_{t_2}\xi\rangle}\right)^{\frac{-h}{2}} =$$
$$= \left(\frac{1+e^{2t_2}\xi_x^2+e^{-2t_2}\xi_y^2}{1+e^{2t}\xi_x^2+e^{-2t}\xi_y^2}\right)^{\frac{h}{4}} \leq \left(\frac{1+2e^{2t_2}\xi_x^2}{e^{2t}\xi_x^2}\right)^{\frac{h}{4}} \leq \left(\frac{4e^{2t_2}\xi_x^2}{e^{2t}\xi_x^2}\right)^{\frac{h}{4}} \leq C_3 e^{-(t-t_2)\frac{h}{2}},$$

where we used $\langle g_t\xi\rangle \geq \langle g_{t_2}\xi\rangle$, $m(g_t\xi) \leq m(g_{t_2}\xi)$, $e^{t_2}\xi \geq e^{-t_2}\xi$ and $m(g_t(\xi)) \leq -\frac{h}{2}$ from the inequality (3) above.

In conclusion,

$$\frac{a_m \circ g_t(x,\xi)}{a_m(x,\xi)} \leq C_1C_2C_3 e^{-t_1\frac{v}{2}} e^{-\frac{t_2-t_1}{2}\min\{v,h\}} e^{-(t-t_2)\frac{h}{2}} \leq Ce^{-\frac{1}{2}\min\{v,h\}t}.$$

□

Appendix B. Anisotropic Sobolev Trace and Extension Theorems

The purpose of this appendix is to prove anisotropic Sobolev trace and extension theorems in $\mathbb{R}^2$ and for open rectangles in $\mathbb{R}^2$. The ideas behind their proofs are standard, but we could not find trace and extension theorems that are well-adapted for our anisotropic norms. The closest reference that we could find is **[LS96]**, but in that paper, the authors restrict to variable-order Sobolev spaces that are positive index, which is not enough for our purposes.

B.1. **Anisotropic Trace Theorem.** We will begin with the following elementary lemma:

Lemma B.1. *Let $h, \sigma, v > 0$, such that $\sigma > h + 1$. Assume $m(\xi_x, \xi_y)$ is a smooth function on $\mathbb{R}^2$ such that $m(\xi_x, \xi_y) = 0$ for $|\xi| < 1/2$, $m(\xi_x, \xi_y) > 0$ for $1/2 \le |\xi| < 1$, and $m(\xi_x, \xi_y) = m(\theta)$ for $|\xi| > 1$, where $m(\theta)$ is smooth in θ and satisfies $m(\theta) = -h$ for $\theta \in (-\epsilon, \epsilon) \cup (\pi - \epsilon, \pi + \epsilon)$, $m(\theta) = v$ for $\theta \in (\pi/2 - \delta, \pi/2 + \delta) \cup (3\pi/2 - \delta, 3\pi/2 + \delta)$, and $-h < m(\theta) < v$ elsewhere, with the constants $\epsilon, \delta > 0$ satisfying $\epsilon + \delta < \pi/2$. Then the function*

$$g(\xi_x, \xi_y) = \left(1 + \xi_x^2\right)^{-\sigma} \left(1 + \xi_x^2 + \xi_y^2\right)^{-m(\xi_x, \xi_y)}$$

belongs to $L^1(\mathbb{R}^2)$.

Proof. We analyze the integral

$$I = \iint_{\mathbb{R}^2} (1 + \xi_x^2)^{-\sigma} (1 + \xi_x^2 + \xi_y^2)^{-m(\xi_x, \xi_y)} \, d\xi_x \, d\xi_y.$$

Using polar coordinates $(\xi_x, \xi_y) = (r\cos\theta, r\sin\theta)$, where the area element transforms as $d\xi_x d\xi_y = r\, dr\, d\theta$, we rewrite the integral as

$$I = \int_0^\infty \int_0^{2\pi} (1 + r^2\cos^2\theta)^{-\sigma} (1 + r^2)^{-m(\theta)} r \, d\theta \, dr.$$

Since $g(\xi_x, \xi_y)$ is bounded in the compact region where $|\xi| < 1$, the integral over this region is finite. Therefore, we focus on the asymptotic behavior as $r \to \infty$, which determines whether $g(\xi_x, \xi_y) \in L^1(\mathbb{R}^2)$. The analysis is carried out separately in three angular sectors:

- In the conical neighborhood $\mathcal{C}_x$ near the ξ_x-axis, where $|\tan\theta| < \epsilon$, we have $m(\theta) = -h$.
- In the conical neighborhood $\mathcal{C}_y$ near the ξ_y-axis, where $|\tan(\theta - \pi/2)| < \delta$, we have $m(\theta) = v$.
- In the intermediate region $\mathcal{I}$, we have $-h < m(\theta) < v$.

For large r, the dominant term in the exponent controls the decay of the integrand. The key step is reducing the integral to the radial term by analyzing the behavior of $\cos^2\theta$ in each sector.

In $\mathcal{C}_x$, the bound $|\cos\theta| \ge 1/\sqrt{1 + \epsilon^2}$ ensures that $1 + r^2\cos^2\theta$ satisfies

$$1 + r^2\cos^2\theta \ge (1 + r^2)(1 - C\theta^2)$$

for some $C > 0$. This allows us to estimate

$$(1 + r^2\cos^2\theta)^{-\sigma} \le (1 + r^2)^{-\sigma}(1 - C\theta^2)^{-\sigma}.$$

Expanding using the inequality $(1 - C\theta^2)^{-\sigma} \le 1 + \sigma C\theta^2$, we obtain

$$(1 + r^2\cos^2\theta)^{-\sigma} \le (1 + r^2)^{-\sigma}(1 + \sigma C\theta^2).$$

Since $m(\theta) = -h$ in this region, the integral simplifies to

$$I_{\mathcal{C}_x} = \int_1^\infty \int_{-\epsilon}^{\epsilon} (1 + r^2)^{h - \sigma}(1 + \sigma C\theta^2) r \, d\theta \, dr.$$

The angular integral is finite, contributing a multiplicative (universal) constant. The condition for convergence reduces to the radial integral

$$\int_1^\infty (1+r^2)^{-h-\sigma} r\,dr,$$

which converges if $h-\sigma<-1$, leading to the condition

$$\sigma - h > 1.$$

In $\mathcal{C}_y$, the bound $|\cos\theta| \le \delta/\sqrt{1+\delta^2}$ ensures that $1+r^2\cos^2\theta$ satisfies

$$1+r^2\cos^2\theta \le (1+r^2).$$

Since $m(\theta)=v$, we estimate

$$(1+r^2\cos^2\theta)^{-\sigma} \le (1+r^2)^{-\sigma}.$$

This reduces the integral to

$$I_{\mathcal{C}_y} = \int_1^\infty \int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}+\delta} (1+r^2)^{-v-\sigma} r\,d\theta\,dr.$$

The angular integral is finite, so the condition for convergence reduces to the radial integral

$$\int_1^\infty (1+r^2)^{-v-\sigma} r\,dr.$$

This integral converges if $-v-\sigma<-1$, leading to the condition

$$\sigma + v > 1,$$

which is satisfied if $\sigma > 1+h$, since $h>0$ and $v>0$.

In the intermediate region $\mathcal{I}$, where $-h<m(\theta)<v$, we interpolate between the previous cases. The worst-case decay determines the necessary conditions, so the same conditions $h+\sigma>1$ and $\sigma+v>1$ are required.

Since all angular sectors impose the same constraints, we conclude that $g(\xi_x,\xi_y) \in L^1(\mathbb{R}^2)$ if and only if $h+\sigma>1$, and this completes the proof. □

We are now ready to prove the anisotropic trace theorem in $\mathbb{R}^2$.

Theorem B.2. *Let $h, v > 1/2$ and let $\sigma > h+1$. Then the restriction operator*

$$\tau_0 \colon W_\omega^{v,-h}(\mathbb{R}^2) \to W_\omega^{-\sigma}(\mathbb{R})$$

defined by

$$\tau_0(f) = f(\cdot, 0)$$

is continuous.

Proof. By density, it suffices to consider $f \in C_\omega^\infty(\mathbb{R}^2)$. The Fourier transform in the x-variable satisfies

$$\mathcal{F}_x\{\tau_0(f)\} = \mathcal{F}(f)(\xi_x, \xi_y = 0),$$

where $\mathcal{F}(f)$ is the full Fourier transform of f. The norm of $\tau_0(f)$ in $W_\omega^{-\sigma}(\mathbb{R})$ is

$$\|\tau_0(f)\|_{W_\omega^{-\sigma}(\mathbb{R})} = \|(1+\xi_x^2)^{-\sigma/2}\mathcal{F}_x\{\tau_0(f)\}\|_{L^2(\mathbb{R})}.$$

Substituting the expression for $\mathcal{F}_x\{\tau_0(f)\}$,

$$\|\tau_0(f)\|_{W_\omega^{-\sigma}(\mathbb{R})} = \|(1+\xi_x^2)^{-\sigma/2}\mathcal{F}(f)(\xi_x,0)\|_{L^2(\mathbb{R})}.$$

Multiplying and dividing by the weight $(1+|\xi|^2)^{m(\xi_x,\xi_y)/2}$, where $m(\xi_x,\xi_y)$ satisfies the conditions from the technical lemma, we rewrite

$$|(1+\xi_x^2)^{-\sigma/2}\mathcal{F}(f)(\xi_x,0)|$$

as

$$|(1+\xi_x^2)^{-\sigma/2}(1+|\xi|^2)^{-m(\xi_x,\xi_y)/2}(1+|\xi|^2)^{m(\xi_x,\xi_y)/2}\mathcal{F}(f)(\xi_x,0)|.$$

Applying the Cauchy-Schwarz inequality in ξ_y, we obtain the bound

$$\left(\int_{\mathbb{R}}(1+\xi_x^2)^{-\sigma}(1+|\xi|^2)^{-m(\xi_x,\xi_y)}\,d\xi_y\right)^{1/2}\left(\int_{\mathbb{R}}(1+|\xi|^2)^{m(\xi_x,\xi_y)}|\mathcal{F}(f)(\xi_x,\xi_y)|^2\,d\xi_y\right)^{1/2}.$$

The first integral corresponds to the integrability of

$$g(\xi_x,\xi_y) = (1+\xi_x^2)^{-\sigma}(1+|\xi|^2)^{-m(\xi_x,\xi_y)}.$$

By Lemma B.1, $g(\xi_x,\xi_y)\in L^1(\mathbb{R}^2)$ if and only if

$$\sigma > 1+h,$$

which is our standing assumption. The second integral in the Cauchy-Schwarz step is controlled by

$$\int_{\mathbb{R}}(1+|\xi|^2)^{m(\xi_x,\xi_y)}|\mathcal{F}(f)(\xi_x,\xi_y)|^2\,d\xi_y \le \|f\|^2_{W_\omega^{v,-h}(\mathbb{R}^2)}.$$

Since the first term is finite by Lemma B.1, integrating over ξ_x gives

$$\|\tau_0(f)\|^2_{W_\omega^{-\sigma}(\mathbb{R})} \le C\|f\|^2_{W_\omega^{v,-h}(\mathbb{R}^2)}.$$

This establishes the continuity of τ_0. □

B.2. **Anisotropic Extension Theorem.** To prove an anisotropic extension theorem, we will first need the following technical lemma,

Lemma B.3. *Let $\phi\in C_0^\infty(\mathbb{R})$ be a smooth cutoff function such that $\phi\equiv 1$ near 0. For any $h>1/2$ and $v>1/2$, there exists a constant $C>0$ such that*

$$\int_{\mathbb{R}}(1+\xi_x^2+\xi_y^2)^{m(\xi_x,\xi_y)}|\phi((1+\xi_x^2)^{1/2}\xi_y)|^2\,d\xi_y \le C(1+\xi_x^2)^{-h}.$$

Proof. Since ϕ is compactly supported, there exists a constant $M>0$ such that $\phi(\eta)\ne 0$ only when

$$|\eta|\le M.$$

Introduce the change of variables

$$\eta = (1+\xi_x^2)^{1/2}\xi_y, \quad \text{so that} \quad d\xi_y = \frac{d\eta}{(1+\xi_x^2)^{1/2}}.$$

Rewriting the integral in terms of η, we obtain

$$I(\xi_x) = \int_{-M}^{M}(1+\xi_x^2+(\eta/(1+\xi_x^2)^{1/2})^2)^{m(\xi_x,\eta/(1+\xi_x^2)^{1/2})}|\phi(\eta)|^2\frac{d\eta}{(1+\xi_x^2)^{1/2}}.$$

Since $m(\xi_x, \xi_y) = -h$ in a neighborhood of the ξ_x-axis, we estimate

$$(1 + \xi_x^2 + (\eta/(1+\xi_x^2)^{1/2})^2)^{m(\xi_x, \eta/(1+\xi_x^2)^{1/2})} \leq (1+\xi_x^2)^{-h}.$$

Thus,

$$I(\xi_x) \leq (1+\xi_x^2)^{-h} \int_{-M}^{M} |\phi(\eta)|^2 d\eta.$$

Since ϕ is a fixed function, the integral is a constant $C := C_\phi$, so

$$I(\xi_x) \leq C(1+\xi_x^2)^{-h-1/2}.$$

Now consider the integral in the region near the ξ_y-axis, where $m(\xi_x, \xi_y) = v$. We must ensure integrability of

$$\int_{\mathbb{R}} (1 + \xi_x^2 + \xi_y^2)^v |\phi((1+\xi_x^2)^{1/2}\xi_y)|^2 \, d\xi_y.$$

For large ξ_y, the term $(1+\xi_y^2)^v$ behaves like ξ_y^{2v}. Since $\phi((1+\xi_x^2)^{1/2}\xi_y)$ ensures ξ_y is bounded up to a scale set by $(1+\xi_x^2)^{-1/2}$, we change variables again:

$$\eta = (1+\xi_x^2)^{1/2}\xi_y, \quad \text{so that} \quad d\xi_y = \frac{d\eta}{(1+\xi_x^2)^{1/2}}.$$

Rewriting the integral,

$$I_{\mathcal{C}_y} = \int_{-M}^{M} (1 + \eta^2/(1+\xi_x^2))^v |\phi(\eta)|^2 \frac{d\eta}{(1+\xi_x^2)^{1/2}}.$$

For large η, the term $(1+\eta^2/(1+\xi_x^2))^v$ behaves like η^{2v}. Thus, the integral

$$\int_{-M}^{M} \eta^{2v} \, d\eta$$

converges if and only if $2v - 1 < -1$, which gives $v > 1/2$. □

We are now ready to state and prove the extension theorem in $\mathbb{R}^2$:

Theorem B.4. *Let $h > 1/2$ and $v > 1/2$. Let $\phi \in C_0^\infty(\mathbb{R})$ be a smooth cutoff function such that $\phi \equiv 1$ near 0. Define the extension operator $E \colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}^2)$ by*

$$(Eu)(x,y) = \mathcal{F}^{-1}_{\xi_x \to x}\big(\phi((1+\xi_x^2)^{1/2}y)\mathcal{F}_{x\to\xi_x}u\big).$$

Then E extends continuously to a bounded operator

$$E \colon W^{-h}_\omega(\mathbb{R}) \to W^{v,-h}_\omega(\mathbb{R}^2),$$

and satisfies $\tau_0(Eu) = u$ for all $u \in W^{-h}_\omega(\mathbb{R})$.

Proof. Let $u \in \mathcal{S}(\mathbb{R})$ and define $f = Eu$. By definition,

$$f(x,y) = \mathcal{F}^{-1}_{\xi_x \to x}\big(\phi((1+\xi_x^2)^{1/2}y)\mathcal{F}_{x\to\xi_x}u\big).$$

Applying the trace operator τ_0, which corresponds to evaluating f at $y = 0$, we obtain

$$\tau_0(f)(x) = f(x,0) = \mathcal{F}^{-1}_{\xi_x \to x}\big(\phi(0)\mathcal{F}_{x\to\xi_x}u\big).$$

Since $\phi(0) = 1$, this simplifies to

$$\tau_0(f)(x) = \mathcal{F}^{-1}_{\xi_x \to x} \mathcal{F}_{x \to \xi_x} u = u(x).$$

Thus, $\tau_0(Eu) = u$ for all $u \in \mathcal{S}(\mathbb{R})$.

To verify that E extends continuously, we analyze the norm of f in $W^{v,-h}_\omega(\mathbb{R}^2)$. The Fourier transform of f is

$$\widehat{f}(\xi_x, \xi_y) = \phi((1 + \xi_x^2)^{1/2} \xi_y) \widehat{u}(\xi_x).$$

Thus, the norm of f in $W^{v,-h}_\omega(\mathbb{R}^2)$ satisfies

$$\|f\|^2_{W^{v,-h}_\omega(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1 + \xi_x^2 + \xi_y^2)^{m(\xi_x, \xi_y)} |\widehat{f}(\xi_x, \xi_y)|^2 \, d\xi_x \, d\xi_y.$$

For each fixed ξ_x, consider the integral in ξ_y:

$$I(\xi_x) = \int_{\mathbb{R}} (1 + \xi_x^2 + \xi_y^2)^{m(\xi_x, \xi_y)} |\phi((1 + \xi_x^2)^{1/2} \xi_y)|^2 \, d\xi_y.$$

By Lemma B.3, we have the estimate

$$I(\xi_x) \leq C(1 + \xi_x^2)^{-h-1/2}.$$

Substituting this bound into the original norm estimate, we obtain

$$\|f\|^2_{W^{v,-h}_\omega(\mathbb{R}^2)} \leq C \int_{\mathbb{R}} (1 + \xi_x^2)^{-h-1/2} |\widehat{u}(\xi_x)|^2 \, d\xi_x.$$

Since $u \in W^{-h}_\omega(\mathbb{R})$,

$$\|f\|^2_{W^{v,-h}_\omega(\mathbb{R}^2)} \leq C \|u\|^2_{W^{-h}_\omega(\mathbb{R})}.$$

This ensures that the extension operator is continuous. $\square$

B.3. **Reduction to Rectangles.** The application of Sobolev trace and extension theorems to rectangles follows by the following standard result, adapted for our anisotropic norms:

Theorem B.5. *Let $h > 1/2$ and $v > 1/2$. Let $R_{a,b} = (0, a) \times (-b, b)$ be an open rectangle in $\mathbb{R}^2$. Then there exists a linear extension operator $\mathcal{R}_{a,b} : W^{v,-h}_\omega(R_{a,b}) \to W^{v,-h}_\omega(\mathbb{R}^2)$ such that:*

$$\|\mathcal{R}_{a,b} f\|_{W^{v,-h}_\omega(\mathbb{R}^2)} \leq C \|f\|_{W^{v,-h}_\omega(R_{a,b})},$$

where $C > 0$ is a constant independent of $f \in W^{v,-h}_\omega(R_{a,b})$.

The construction of the extension is standard, and holds more generally for all domains satisfying the so-called finite cone property, which is satisfied by $R_{a,b}$. Moreover, the extension operator $\mathcal{R}_{a,b}$ in our case can be made explicit: one systematically extends $f(x, y)$ beyond $R_{a,b}$ using smooth reflections and cutoff functions. We omit the proof of this theorem, as well as the tedious expression of the extension operator $\mathcal{R}_{a,b}$.

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