A CENTRAL LIMIT THEOREM FOR THE KONTSEVICH-ZORICH COCYCLE

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Abstract. We show that a central limit theorem holds for exterior powers of the Kontsevich-Zorich (KZ) cocycle. In particular, we show that, under the hypothesis that the top Lyapunov exponent on the exterior power is simple, a central limit theorem holds for the lift of the (leafwise) hyperbolic Brownian motion to any strongly irreducible, symplectic, $\mathrm{SL}(2,\mathbb{R})$-invariant subbundle, that is moreover symplectic-orthogonal to the so-called tautological subbundle. We then show that this implies that a central limit theorem holds for the lift of the Teichmüller geodesic flow to the same bundle.

For the random cocycle over the hyperbolic Brownian motion, we are able to prove under the same hypotheses that the variance of the top exponent is strictly positive. For the deterministic cocycle over the Teichmüller geodesic flow we can prove that the variance is strictly positive only for the top exponent of the first exterior power (the KZ cocycle itself) under the hypothesis that its Lyapunov spectrum is simple.

1. Introduction

This paper concerns the Kontsevich-Zorich (KZ) cocycle, a much studied dynamical system in the field of Teichmüller dynamics. The KZ cocycle has played a major role in addressing multiple questions of physical interest, ranging from illumination problems **[LMW16]** to the computation of diffusion rates on wind-tree models **[DHL14]**, and it itself acts as a renormalizing dynamical system for straight-line flows on translation surfaces. We refer the reader to the surveys **[Zor06, FM14, Wri15]** for an introduction to this rich area of research.

It is now well-established that Hodge theory, together with classical potential theory, can be brought to bear on this cocycle and its associated Lyapunov exponents, thanks to the pioneering works of M. Kontsevich and A. Zorich **[KZ97, Kon97]**, later developed in **[For02]**. In these works, the Hodge norm was introduced in Teichmüller dynamics, and in **[For02]** it was proved that the logarithm of the Hodge norm is a subharmonic function on all exterior powers of the cocycle, hence the KZ cocycle has positive exponents on strata (see also **[For06]**, **[For11]**, **[FMZ11]**, **[FMZ12]**).

The Hodge norm has since played a crucial role in the developments in Teichmüller dynamics, in particular in the study of the hyperbolicity properties of the KZ cocycle and of the Teichmüller flow, and related questions in the ergodic theory of translations flows and interval exchange transformations. A very partial list of landmark applications of the Hodge norm includes [**ABEM12, EKZ14, EMM15, EM18, Fil16a, Fil16b, Fil17**].

The purpose of this paper is to show that probabilistic potential theory (and thus stochastic calculus), can be applied to study the oscillations of the Hodge norm of the KZ cocycle. In fact, we prove a (non-commutative) central limit theorem (CLT) for exterior powers of both the random and the deterministic KZ cocycles, and we moreover prove the non-degeneracy of the CLT for the random cocycles and for the first exterior power of the deterministic KZ cocycle (the KZ cocycle itself) under the natural dynamical assumptions of simplicity of the Lyapunov spectrum. Motivated by computer experiments and the results in [**Zor96, Zor97**], the simplicity of the KZ spectrum for all strata of the moduli space of Abelian differentials was conjectured by M. Kontsevich and A. Zorich in [**KZ97, Kon97**]. It was then established by A. Avila and M. Viana in [**AV07**], and, in genus 2, for all $\mathrm{SL}(2,\mathbb{R})$-invariant orbifolds, by M. Bainbridge in [**Bai07**].

The problem of finding oscillations of the KZ cocycle has been the subject of recent interest: in [**AS21b**], a mechanism to produce oscillations of the KZ cocycle was presented, where the basepoint is a fixed surface, and a more refined mechanism was developed by J. Chaika, O. Khalil and J. Smillie in their work on the ergodic measures of the Teichmüller horocycle flow [**CKS21**]. We expect that the deterministic central limit theorem presented here can be brought to bear on the scope of these results.

The probabilistic ideas that inspired our approach, and their application to geodesic flows, go back to the work of Y. Le Jan [**LJ94**], and we refer the reader to [**FLJ12**] for both an introduction to stochastic calculus and to the remarkable ideas that appear in [**LJ94**].

The approach we follow to prove the CLT for the random cocycle also relies on the analysis of $\mathrm{SL}(2,\mathbb{R})$ unitary representations to solve a (leafwise) Poisson equation, and leverages crucially the spectral gap of the leafwise hyperbolic Laplacian, which is due to A. Avila, S. Gouëzel and J.-C. Yoccoz [**AGY06**] and A. Avila and S. Gouëzel [**AG13**].

The CLT for the deterministic cocycle is then derived from the corresponding result for the random cocycle by a stopping time argument based in part on an asymptotic estimate due to A. Ancona [**Anc90**].

We point out that in the setting of products of independent and identically distributed random matrices, the central limit theorem was established, in varying levels of generality, by Bellman in [**Bel54**], H. Furstenberg and H. Kesten in [**FK60**], Tutubalin in [**Tut77**], Le Page in [**LP82**], Y. Guivarc'h and A. Raugi in [**GR85**], I. Ya. Golsheid and G. A. Margulis in [**GM89**], Hennion in [**Hen97**], Jan in [**Jan00**], and more recently, and under an optimal finite second moment condition, by Y. Benoist and J.-F. Quint in [**BQ16**]. The central limit theorem was also established for solutions of linear stochastic differential equations with Markovian coefficients by P. Bougerol in [**Bou88**]. On the other hand, to the best of our knowledge, there are no comparable works in the setting of deterministic cocycles over (non-uniformly) hyperbolic flows, such as the one we treat here. We point to **[DKP21]**, **[FK21]**, **[PP22]** for some results on the central limit theorem in this direction. While not the original aim of this paper, we note that our work addresses, if ever so incrementally, this dearth of central limit theorem results for deterministic cocycles over hyperbolic flows (a related result for the KZ cocycle, based on the study of a transfer operator via anisotropic Banach spaces, has been recently announced by O. Khalil).

In another direction, the paper of J. Daniels and B. Deroin **[DD19]** adapted the Teichmüller dynamics methodology to more general compact Kähler manifolds, and one in which the methods in this paper are applicable, provided that we can prove existence of a solution to Poisson's equation for the corresponding Laplacian.

In **[DFV17]**, D. Dolgopyat, B. Fayad and I. Vinogradov proved a central limit theorem for the Siegel transform of sufficiently regular observables for the diagonal action on the space of lattices. Their methods are in fact much more general and imply in particular a Central Limit Theorem for pushforwards of (unstable) unipotent arcs with respect to the uniform distribution on almost every unipotent orbit [**DFV17**, Theorem 7.1, Corollary 7.2]. It would be interesting to prove exponential mixing for the action of the Teichmüller flow on the projectivized Hodge bundle, with the aim of applying a multiplicative generalization of their results to the KZ cocycle.

2. Statement of results

Let $\pi : \mathbb{P}(\mathbf{H}) \to X$ be the projectivization of a strongly irreducible $\mathrm{SL}(2,\mathbb{R})$-invariant symplectic subbundle of the absolute (real) Hodge bundle over an $\mathrm{SL}(2,\mathbb{R})$ orbit closure X, whose fiber over each point in X is $H^1(S,\mathbb{R})$, and with ν an ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure on X. The Kontsevich-Zorich cocycle is the lift of the $\mathrm{SL}(2,\mathbb{R})$ action to $\mathbb{P}(\mathbf{H})$, obtained by parallel transport with respect to the Gauss-Manin connection. Furthermore, the cocycle acts symplectically since it preserves the intersection form on $H^1(S,\mathbb{R})$.

For our purposes, we will be concerned with k-th exterior powers $\mathbf{H}^{(k)}$ of strongly irreducible invariant symplectic components $\mathbf{H}$ of the Hodge bundle, which are symplectic orthogonal to the tautological subbundle (spanned for every $\omega \in X$ by $[\mathrm{Re}\,\omega]$ and $[\mathrm{Im}\,\omega]$). We will denote by $\mathbb{P}(\mathbf{H}^{(k)})$ the projectivization of the bundle $\mathbf{H}^{(k)}$. This bundle supports an $\mathrm{SO}(2,\mathbb{R})$-invariant probability measure $\hat{\nu}$ such that, for ν-a.e ω, the conditional measure on $\mathbb{P}(\mathbf{H}^{(k)}_\omega)$ is the Haar measure. An Euclidean structure is in fact given by the Hodge norm (see 3.5 for the definition), and which we use in the sequel.

Therefore, we fix the $\mathrm{SO}(2,\mathbb{R})$-invariant Hodge norm $\|\cdot\|^{(k)}_{\pi(\cdot)}$ on $\mathbb{P}(\mathbf{H}^{(k)})$. Define $\sigma_k : \mathrm{SL}(2,\mathbb{R}) \times \mathbb{P}(\mathbf{H}^{(k)}) \to \mathbb{R}$ by

$$\sigma_k(g,\mathbf{v}) = \log \frac{\|g\mathbf{v}\|^{(k)}_{g\pi(\mathbf{v})}}{\|\mathbf{v}\|^{(k)}_{\pi(\mathbf{v})}}.$$

Let $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ denote the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$, whose action on the orbit closure X yields the Teichmüller flow. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h$ denote the non-negative Lyapunov exponents of the Kontsevich–Zorich cocycle on a symplectic strongly irreducible subbundle $\mathbf{H}$ of dimension $2h \in \{2, \dots, 2g\}$. Since the cocycle is symplectic on $\mathbf{H}$, the top h exponents determine the entire Lyapunov spectrum.

For $\omega \in X$, let $\mathbf{v}_\omega$ in $\mathbb{P}(\mathbf{H}^{(k)}_\omega)$ be any k-dimensional exterior vector (of dimension $k \leq h$) in $\mathbf{H}_\omega$. For $\hat{\nu}$-a.e. $\mathbf{v} = (\omega, \mathbf{v}_\omega)$, it is a consequence of the multiplicative ergodic theorem that

$$\lim_{T\to\infty} \frac{\sigma_k(g_T,\mathbf{v})}{T} = \sum_{i=1}^{k} \lambda_i \,.$$

Our main result is the following:

Theorem 2.1. *Let $\mathbf{H}$ be a strongly irreducible, symplectic, $\mathrm{SL}(2,\mathbb{R})$-invariant subbundle, which is symplectic orthogonal to the tautological subbundle. If $\lambda_k > \lambda_{k+1}$, then there exists a real number $V^{(k)}_{g_\infty} \geq 0$ such that*

$$\lim_{T\to\infty} \hat{\nu}\left(\left\{\mathbf{v} \in \mathbb{P}(\mathbf{H}^{(k)}) \ : \ a \leq \frac{1}{\sqrt{T}}\left(\sigma_k(g_T,\mathbf{v}) - T(\sum_{i=1}^{k}\lambda_i)\right) \leq b\right\}\right)$$
$$= \frac{1}{\sqrt{2\pi V^{(k)}_{g_\infty}}} \int_a^b \exp(-x^2/V^{(k)}_{g_\infty})dx.$$

Moreover, if the Lyapunov spectrum is simple, then $V^{(1)}_{g_\infty} > 0$.

Remark 2.2. The statement also holds in the event that $V^{(k)}_{g_\infty} = 0$, and in that case the resulting distribution would be a delta distribution. The positivity of the variance holds for 2-dimensional subbundles with strictly positive top Lyapunov exponent (for instance on the symplectic orthogonal of the tautological subbundle in genus 2 for any $\mathrm{SL}(2,\mathbb{R})$-invariant measure), as in this case the simplicity condition on the top exponent is trivially satisfied.

Remark 2.3. The simplicity of the Lyapunov spectrum is established for the canonical Masur-Veech measures on strata by A. Avila and M. Viana in [**AV07**], and we remark that, in genus 2, this is established for all ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measures by M. Bainbridge in [**Bai07**].

Remark 2.4. The assumption that $\mathbf{H}$ is symplectic orthogonal to the tautological subbundle precludes the equality $2h = 2g$, which in turn precludes the equality $k = g$. It also follows by the spectral gap property of the Kontsevich-Zorich cocycle that for any g_t-invariant and ergodic probability measure, $\lambda_1 < 1$ [**For02**] (see also [**FMZ12**, Corollary 2.2]).

To prove Theorem 2.1, we will first work with the hyperbolic Brownian motion, which is the diffusion process generated by the foliated hyperbolic Laplacian. Let ρ be a (foliated) hyperbolic Brownian motion trajectory starting at a (generic) basepoint $\omega \in X$, defined almost everywhere with respect to a probability measure $\mathbb{P}_\omega$ on the space of such trajectories W_ω. This process is in fact defined on $X^* = \mathrm{SO}(2,\mathbb{R})\backslash X$. Moreover, ρ can be lifted to $\mathrm{SL}(2,\mathbb{R})$, and is moreover defined by taking the outward radial unit tangent vector at all points. We continue to refer to the lifted path as ρ by abuse of notation. Additionally, the space X gives rise to a product space $X^W := X \otimes W$ whose fiber over each point ω in X is W_ω, and which also supports a measure $\nu_{\mathbb{P}} := \nu \otimes \mathbb{P}$, whose conditional measure over a point ω is $\mathbb{P}_\omega$. We can thus similarly define the product W-Hodge bundle $\mathbb{P}^W(\mathbf{H}^{(k)})$, whose fiber over each point (ω, ρ) in X^W is $\mathbf{H}^{(k)}_\omega$. A pair $(\rho, \mathbf{v}) \in \mathbb{P}^W(\mathbf{H}^{(k)})$ is thus defined to be the lift of the path ρ (starting at ω) to $\mathbb{P}^W(\mathbf{H}^{(k)})$, obtained by parallel transport with respect to the Gauss-Manin connection. This in turn would also give rise to a measure $\hat{\nu}_{\mathbb{P}} := \hat{\nu} \otimes \mathbb{P}$ whose conditional measure over a point $\mathbf{v}$ is $\mathbb{P}_\omega$. We therefore also have

Theorem 2.5. *Let $\mathbf{H}$ be a strongly irreducible, symplectic, $\mathrm{SL}(2,\mathbb{R})$-invariant subbundle, which is symplectic orthogonal to the tautological subbundle. If $\lambda_k > \lambda_{k+1}$, then there exists a real number $V^{(k)}_{\rho_\infty} > 0$ such that*

$$\lim_{T\to\infty} \hat{\nu}_{\mathbb{P}} \left(\left\{ (\rho, \mathbf{v}) \in \mathbb{P}^W(\mathbf{H}^{(k)}) \ : \ a \leq \frac{1}{\sqrt{T}} \left(\sigma_k(\rho_T, \mathbf{v}) - T(\sum_{i=1}^{k} \lambda_i) \right) \leq b \right\} \right)$$
$$= \frac{1}{\sqrt{2\pi V^{(k)}_{\rho_\infty}}} \int_a^b \exp(-x^2/V^{(k)}_{\rho_\infty}) dx.$$

Remark 2.6. Observe that for $g = 2$, the symplectic orthogonal bundle to the tautological bundle has dimension 2, hence it is strongly irreducible. Our two results reduce to ones that concern the second Lyapunov exponent of the Kontsevich-Zorich cocycle on the full Hodge bundle.

In addition to the Hodge theoretic techniques that we employ, some ingredients of our proof include

- results of Avila-Gouëzel-Yoccoz [**AGY06**] and Avila-Gouëzel [**AG13**] on the spectral gap of the leafwise hyperbolic Laplacian, together with the analysis of $\mathrm{SL}(2,\mathbb{R})$ unitary representations (as in Flaminio-Forni [**FF03**]), to show existence of a unique zero-average solution of a Poisson equation (see Appendix A);

- elementary stochastic calculus to extract and control the necessary oscillations;
- and an asymptotic estimate due to Ancona [**Anc90**] to relate the geodesic flow with the Brownian motion.

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This paper builds on and extends thesis work of H. A. [**AS21a**] (written under the direction of G. F.), which dealt with the genus 2 case and more generally the top exterior power of the Kontsevich-Zorich cocycle, but left unanswered the positivity of the deterministic variance question and extensions to lower exterior powers that are addressed in the present paper. H. A. is grateful to G. F. for suggesting this fruitful direction of research, for his patience and encouragement, and for invaluable conversations and ideas on this subject. H. A. is grateful to Jon Chaika, Dima Dolgopyat, and Bassam Fayad for their interest in his thesis project and for insightful and enlightening conversations. H. A. acknowledges support from the NSF grants DMS 1600687, DMS 1956049, and DMS 2101464. G. F. acknowledges support from the NSF grants DMS 1600687 and DMS 2154208.

3. Preliminaries

3.1. **Translation surfaces.** Let S be a Riemann surface of genus $g \geq 2$, and ω a holomorphic 1-form on S. The pair (S, ω) is called a *translation surface*, since ω induces an atlas whose coordinate changes are translations on $\mathbb{C} \equiv \mathbb{R}^2$. In other terms, ω gives a flat metric with finitely many conical singularities and trivial holonomy on S, and the zero set of ω characterizes the singularity set of the conical metric. The area of a translation surface is given by $\int_S \omega \wedge \overline{\omega}$. We will refer to the pair (S, ω) as just ω.

3.2. **Moduli Space.** Let $\mathcal{TH}_g$ be the Teichmüller space of unit-area translation surfaces of genus $g \geq 2$, and let $\mathcal{H}_g = \mathcal{TH}_g/\mathrm{Mod}_g$ be the corresponding moduli space, where Mod_g denotes the mapping class group. The space $\mathcal{H}_g$ is partitioned into strata $\mathcal{H}_\kappa$, which consist of all unit-area translation surfaces whose conical singularities have total angles $2\pi(1+\kappa_1), \dots, 2\pi(1+\kappa_s)$, as $\kappa = (\kappa_1, \dots, \kappa_s)$ varies over multi-indices with $\sum \kappa_i = 2g-2$.

Local period coordinates on each stratum are defined by the map which takes every holomorphic 1-form ω to its cohomology class $[\omega]$ in $H^1(S, \Sigma_\omega, \mathbb{C})$, relative to the set Σ_ω of its zeros. The set of all period coordinate maps defines an affine structure on each stratum, since all changes of coordinates are given by affine maps.

3.3. $\mathrm{SL}(2, \mathbb{R})$ **action.** There is a natural action of $\mathrm{SL}(2, \mathbb{R})$ on the space of all translation surfaces which descends to their Teichmüller and moduli spaces. It is proved in [**EM18, EMM15**] that, for any $\omega \in \mathcal{H}(\kappa)$, the closure X of $\mathrm{SL}(2, \mathbb{R}) \cdot \omega$ is an affine invariant suborbifold, and supports a unique ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure ν in the Lebesgue measure class, given by the normalized Lebesgue measure in period coordinates.

3.4. **Kontsevich-Zorich cocycle.** Let $\widehat{\mathbf{H}}_\kappa(S,\mathbb{R}) = \mathcal{TH}_\kappa \times H^1(S,\mathbb{R})$, and for every $g \in \mathrm{SL}(2,\mathbb{R})$, let $\widehat{g} : \widehat{\mathbf{H}}_\kappa(S,\mathbb{R}) \to \widehat{\mathbf{H}}_\kappa(S,\mathbb{R})$ be the trivial cocycle map defined as

$$\widehat{g}(\omega, c) = (g\omega, c)\,, \qquad \text{for } \omega \in \mathcal{TH}_\kappa \text{ and } c \in H^1(S,\mathbb{R})\,,.$$

The absolute (real) Hodge bundle is given by $\mathbf{H}_\kappa(S,\mathbb{R}) = \widehat{\mathbf{H}}_\kappa(S,\mathbb{R})/\mathrm{Mod}_g$ and the Kontsevich-Zorich cocycle g is the projection of $\widehat{g}$ to $\mathbf{H}_\kappa(S,\mathbb{R})$.

3.5. **Hodge inner product and the second fundamental form.** Given two holomorphic 1-forms ω_1, ω_2 in $\Omega(S)$, where $\Omega(S)$ is the vector space of holomorphic 1-forms on S, the Hodge inner product is given by the formula

$$\langle \omega_1, \omega_2 \rangle := \frac{i}{2} \int_S \omega_1 \wedge \overline{\omega_2}$$

Moreover, the Hodge representation theorem implies that for any given cohomology class $c \in H^1(S,\mathbb{R})$, there is a unique holomorphic 1-form $h(c) \in \Omega(S)$, such that $c = [\mathrm{Re}\, h(c)]$ (cf. [**FMZ12**]). The Hodge inner product for two real cohomology classes $c_1, c_2 \in H^1(S,\mathbb{R})$ is defined as

$$A_\omega(c_1, c_2) := \langle h(c_1), h(c_2) \rangle.$$

The second fundamental form B_ω (of the Gauss-Manin connection with respect to the Chern connection for the holomorphic structure of the Hodge filtration) is defined as

$$B_\omega(c_1, c_2) := \frac{i}{2} \int_S \frac{h(c_1) h(c_2)}{\omega^2} \omega \wedge \overline{\omega}.$$

Let H_ω denote the curvature operator of the second fundamental form.

Remark 3.1. It is known that B_ω vanishes identically in the symplectic orthogonal of the tautological subbundle on only two orbit closures, namely the *Eierlegende Wollmilchsau* and *Ornithorynque*, and this follows from the works [**Aul16, EKZ14, Möl11, AN20**]. By a result of S. Filip [**Fil17**], the rank of the second fundamental form B equals the number of strictly positive Lyapunov exponents of the Kontsevich–Zorich cocycle.

In the following $\mathbf{H}$ will denote a strongly irreducible, symplectic, $\mathrm{SL}(2,\mathbb{R})$-invariant subbundle, which is symplectic orthogonal to the tautological bundle. For any isotropic k-dimensional exterior vector $\mathbf{c}_\omega$ in $\mathbf{H}_\omega^{(k)}$, it also follows by [**For02**] (see also [**FMZ12**, Corollary 2.2]) that

$$\left| \frac{d}{dt} \sigma_k(g_t, \mathbf{c}_\omega) \right| < k \tag{3.5.1}$$

For $h \in \{1, \dots, g-1\}$, let $\{c_1, c_2, \dots, c_h\}$ be a Hodge-orthonormal basis of $\mathbf{H} \subset H^1(S,\mathbb{R})$, and let $A_\omega^{(h)}$ (resp., $B_\omega^{(h)}$) be the corresponding representation matrix of the Hodge inner product A_ω (resp., of the second fundamental form B_ω). Let $H_\omega^{(h)} = B_\omega^{(h)} \bar{B}_\omega^{(h)}$ be the matrix of the curvature operator, which is Hermitian non-negative, since $B_\omega^{(h)}$ is symmetric. The eigenvalues of $B_\omega^{(h)}$ are denoted by $\Lambda_i(\omega)$, where $|\Lambda_1| > |\Lambda_2| \geq \cdots \geq |\Lambda_h| \geq 0$. Moreover, the norm squared of these eigenvalues, $|\Lambda_i(\omega)|^2$, are the eigenvalues of the curvature matrix $H_\omega^{(h)}$, which are continuous, bounded functions on $\mathcal{H}_g$ (cf. **[FMZ12]**, Lemma 2.3).

For any k-dimensional exterior vector $\mathbf{v} \in \mathbb{P}(\mathbf{H}^{(k)})$, let

$$\{c_1, c_2, \ldots, c_k, c_{k+1}, \ldots, c_h\} \subset \mathbf{H}$$

be an ordered orthonormal basis such that $\{c_1, c_2, \ldots, c_k\}$ is a basis of $\mathbf{v}$. We let $A_\omega^{(k)}(\mathbf{v})$ (resp., $B_\omega^{(k)}(\mathbf{v})$) be the corresponding representation matrix of the Hodge inner product A_ω (resp., of the second fundamental form B_ω) restricted to $\mathbf{v}$ with respect to the basis $\{c_1, c_2, \ldots, c_k\}$. We let $H_\omega^{(k)}(\mathbf{v})$ be the representation matrix of the restriction of the curvature operator H_ω to $\mathbf{v}$ with respect to the basis $\{c_1, c_2, \ldots, c_k\}$.

3.6. **Foliated Hyperbolic Laplacian.** The space $\mathcal{H}_g$, is foliated by the orbits of the $\mathrm{SL}(2,\mathbb{R})$-action, whose leaves are isometric to the unit cotangent bundle of the Poincaré disk $\mathbb{D}$. For $\omega \in \mathcal{H}_g$, the Teichmüller disk $L_\omega := \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \cdot \omega$ is isometric to $\mathbb{D}$, and so is endowed with the (foliated) hyperbolic gradient ∇_{L_ω} and hyperbolic Laplacian Δ_{L_ω}. Let $r_\theta = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$.

Remark 3.2. Observe that for $\omega \in X$, the Teichmüller disk L_ω is identified with $\mathbb{D}$ via the map $(t, \theta) \mapsto \mathrm{SO}(2,\mathbb{R}) \cdot g_t r_\theta \omega$.

Now suppose that $f : X \to \mathbb{R}$ is an $\mathrm{SO}(2,\mathbb{R})$-invariant C^∞-function in the direction of the leaf. For $\omega \in X$ and for L_ω the Teichmüller disk passing through ω, we define $\Delta f(\omega) := \Delta_{L_\omega} f|_{L_\omega}(\omega)$, where $f|_{L_\omega}$ is the restriction of f to L_ω. We also define the leafwise gradient similarly.

Observe that the Hodge inner product $A_\omega(\cdot,\cdot)$ is invariant under the action of $\mathrm{SO}(2,\mathbb{R})$, and so defines a real-analytic function on the Teichmüller disk. In the sequel, we will only work in a given Teichmüller disk, so the norm will read $(\cdot,\cdot)_z$ for a complex parameter $z \in \mathbb{D}$. For any k-dimensional exterior vector $\mathbf{v} = (\omega, \mathbf{v}_\omega)$ in the symplectic orthogonal of the tautological subbundle (with the origin $z = 0$ corresponding to ω as in 3.2), define

$$\sigma_k(z, \mathbf{v}) := \log |\det A_z^{(k)}(\mathbf{v})|^{1/2},$$

where $A_z^{(k)}(\mathbf{v}) = A_z(\mathbf{v}_i, \mathbf{v}_j)$ and $\{\mathbf{v}_i\}$ is an ordered basis of $\mathbf{v}$.

Remark 3.3. In fact, this is an abuse of notation since we originally lifted elements of $\mathrm{SL}(2,\mathbb{R})$ to the Hodge bundle. This is not an issue since the Hodge norm is $\mathrm{SO}(2,\mathbb{R})$-invariant.

We recall the following fundamental fact

Theorem 3.4. **[For02, FMZ12]** *For every $1 \leq k \leq h$ there exist smooth functions $\Phi_k : \mathbb{P}(\boldsymbol{H}^{(k)}) \to [0,k]$ and $\Psi_k : \mathbb{P}(\boldsymbol{H}^{(k)}) \to D(0,k) \subset \mathbb{C}$ such that the following holds. For any k-dimensional exterior vector $\mathbf{v} \in \mathbb{P}(\boldsymbol{H}^{(k)})$, we have the following identities:*

$$\Delta_{L_\omega}\sigma_k(z,\mathbf{v}) = 2\Phi_k(z,\mathbf{v}) \quad \textit{and} \quad \nabla_{L_\omega}\sigma_k(z,\mathbf{v}) = \Psi_k(z,\mathbf{v})\,. \tag{3.6.1}$$

In the particular case that $k = h$, there exist functions $\Lambda_i : X \to D(0,1)$ for all $i \in \{1,\ldots,h\}$ such that

$$\Delta_{L_\omega}\sigma_h(z,\mathbf{v}) = 2\sum_{i=1}^{h}|\Lambda_i(z)|^2 \quad \textit{and} \quad \nabla_{L_\omega}\sigma_h(z,\mathbf{v}) = \sum_{i=1}^{h}\Lambda_i(z)\,.$$

In particular, in this case, the Laplacian and the gradient are independent of the choice of a maximal isotropic (Lagrangian) subspace $\mathbf{v} \in \mathbb{P}(\boldsymbol{H}^{(h)})$. Moreover, for all $k \in \{1,\ldots,h\}$, under the condition that $\lambda_k > \lambda_{k+1}$, which implies that the unstable Oseledets isotropic k-dimensional distribution is well-defined, for ν a.e. ω, we have that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \Delta_{L_\omega}\sigma_k(g_t,\mathbf{v})\,dt = \int_X 2\Phi_k(\omega, E_k^+(\omega))d\nu = 2\sum_{i=1}^{k}\lambda_i\,.$$

Remark 3.5. The functions Φ_k and Ψ_k can be written as follows. Let $B_z^{(k)}(\mathbf{v})$ and $H_z^{(k)}(\mathbf{v})$ denote the restrictions of the second fundamental form and of the curvature to the k-dimensional exterior vector $\mathbf{v} \in \mathbb{P}(\mathbf{H}^{(k)})$. By definition $B^{(k)}$ and $H^{(k)}$ are functions on $\mathbb{P}(\mathbf{H}^{(k)})$ with values in the subspace of complex symmetric $k \times k$ matrices and non-negative Hermitian $k \times k$ matrices. The following formulas hold, for all $(\omega,\mathbf{v}) \in \mathbb{P}(\mathbf{H}^{(k)})$:

$$\begin{aligned} \Phi_k(\omega,\mathbf{v}) &= 2\mathrm{tr}(H_\omega^{(k)}(\mathbf{v})) - \mathrm{tr}\left(B_\omega^{(k)}(\mathbf{v})\bar{B}_\omega^{(k)}(\mathbf{v})\right)\,; \\ \Psi_k(\omega,\mathbf{v}) &= \mathrm{tr}(B_\omega^{(k)}(\mathbf{v}))\,. \end{aligned} \tag{3.6.2}$$

3.7. **Harmonic measures.** A probability measure μ on $\mathrm{SO}(2,\mathbb{R})\backslash X$ is called harmonic if for all bounded functions $f : \mathrm{SO}(2,\mathbb{R})\backslash X \to \mathbb{R}$ of class C^∞ in the leaf direction,

$$\int_{\mathrm{SO}(2,\mathbb{R})\backslash X} \Delta f(\omega)\,d\mu = \int_{\mathrm{SO}(2,\mathbb{R})\backslash X} \Delta_{L_\omega} f|_{L_\omega}(\omega)\,d\mu = 0.$$

Such a measure is also ergodic if $\mathrm{SO}(2,\mathbb{R})\backslash X$ cannot be partitioned into two union of leaves, each of which having positive μ measure. We refer the reader to the interesting paper of Lucy Garnett **[Gar83]** for details and for an ergodic theorem for such measures. It is also a fact, due to Bakhtin-Martinez **[BM08]**, that harmonic measures on $\mathrm{SO}(2,\mathbb{R})\backslash X$ are in one-to-one correspondence with P-invariant measures on X. This is closely related to a classical fact due to Furstenberg **[Fur63a, Fur63b]** that P-invariant measures are in one-to-one correspondence with (admissible) stationary measures, and that harmonic measures are stationary. In the case of $\mathrm{SL}(2,\mathbb{R})$, these three notions are therefore closely related.

3.8. **Hyperbolic Brownian Motion.** Following the normalization used in **[For02]** (which is a standard normalization, see also **[Hel00]**), for $z = re^{i\theta}$ with $\theta \in [0, 2\pi]$, write

$$t := \frac{1}{2} \log \frac{1+r}{1-r}. \tag{3.8.1}$$

Since the Hodge norm is $\mathrm{SO}(2,\mathbb{R})$-invariant, it suffices to study the diffusion process generated by $\frac{1}{2}\Delta_{L_\omega}$, where the leafwise hyperbolic Laplacian in geodesic polar coordinates is given by

$$\Delta_{L_\omega} = \frac{\partial^2}{\partial t^2} + 2\coth(2t)\frac{\partial}{\partial t} + \frac{4}{\sinh^2(2t)}\frac{\partial^2}{\partial \theta^2}. \tag{3.8.2}$$

Moreover, let $(W_\omega^{(i)}, \mathbb{P}_\omega^{(i)})$, $i = 1, 2$, be two copies of the space of Brownian trajectories $C(\mathbb{R}^+, \mathbb{R})$ starting at the origin (with the origin corresponding to a random point ω), together with the standard Wiener measure, and such that $W_\omega^{(1)}$ and $W_\omega^{(2)}$ are independent. Set $W_\omega = W_\omega^{(1)} \times W_\omega^{(2)}$ and $\mathbb{P}_\omega = \mathbb{P}_\omega^{(1)} \times \mathbb{P}_\omega^{(2)}$. The hyperbolic Brownian motion is the diffusion process $\rho_s = (t(s), \theta(s))$ generated by the (leafwise) hyperbolic Laplacian. It follows by Ito's formula **[FLJ12**, Theorem VI.5.6] that the generator determines the trajectories of the diffusion process ρ_s which are solutions of the following stochastic differential equations

$$dt(s) = dW_s^{(1)} + \coth(2t(s))ds \tag{3.8.3}$$

$$d\theta(s) = \frac{2}{\sinh(2t(s))} dW_s^{(2)} \tag{3.8.4}$$

with $\rho_0 = 0$.

In addition, for an $\mathrm{SO}(2,\mathbb{R})$-invariant function $f : X \to \mathbb{R}$, where f is of class C^2 along $\mathrm{SL}(2,\mathbb{R})$ orbits, Ito's formula gives

$$f(\rho_T) - f(\rho_0) = \int_0^T \left(\frac{\partial}{\partial t} f(\rho_s), \frac{2}{\sinh(2t(s))}\frac{\partial}{\partial \theta} f(\rho_s)\right) \cdot \left(dW_s^{(1)}, dW_s^{(2)}\right) \tag{3.8.5}$$

$$+ \int_0^T \left(\frac{1}{2}\frac{\partial^2}{\partial t^2} f(\rho_s) + \frac{1}{2} 2\coth(2t(s))\frac{\partial}{\partial t} f(\rho_s) + \frac{1}{2}\frac{4}{\sinh^2(2t(s))}\frac{\partial^2}{\partial \theta^2} f(\rho_s)\right) ds \tag{3.8.6}$$

$$= \int_0^T \nabla_{L_\omega} f(\rho_s) \cdot (dW_s^{(1)}, dW_s^{(2)}) + \frac{1}{2}\int_0^T \Delta_{L_\omega} f(\rho_s) ds\,. \tag{3.8.7}$$

Finally, we note that the foliated heat semigroup D_t is given as follows

$$D_s f(x) := \int_X \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty f(z) p_\omega(t, s) \sinh(t) dt\, d\theta\, d\nu \tag{3.8.8}$$

where $p_\omega(t,s)$ is the (foliated) hyperbolic heat kernel at time s; in other words, for $x, y \in L_\omega$, this is the transition probability kernel $p_\omega(x, y; s)$, with $d_{\mathbb{D}}(x, y) = t$.

4. Proofs of Main Theorems

4.1. **Distributional Convergence in Theorem 2.5.** Recall that ρ_s is the diffusion process generated by the foliated hyperbolic Laplacian. We are interested in studying the term

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_T, \mathbf{v}) - T\sum_{i=1}^{k}\lambda_i). \tag{4.1.1}$$

Set $\lambda_{(k)} = \sum_{i=1}^{k}\lambda_i$. By applying Ito's formula, we obtain,

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_T, \mathbf{v}) - T\lambda_{(k)}) = \frac{\sigma_k(\rho_0, \mathbf{v})}{\sqrt{T}} + \frac{1}{\sqrt{T}}\int_0^T \nabla_{L_\omega}\sigma_k(\rho_s, \mathbf{v})\cdot(dW_s^{(1)}, dW_s^{(2)}) \tag{4.1.2}$$

$$+\frac{1}{2\sqrt{T}}\int_0^T(\Delta_{L_\omega}\sigma_k(\rho_s, \mathbf{v}) - 2\lambda_{(k)})ds \tag{4.1.3}$$

Let $\{X, Y, \Theta\}$ be the standard generators of the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ corresponding to the geodesic flow, the orthogonal geodesic flow and the maximal compact subgroup $\mathrm{SO}(2,\mathbb{R})$, given by the formulas:

$$X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}. \tag{4.1.4}$$

For any functions $U \in L^2(X,\mu)$ and for every $\omega \in X$, let now u_ω denote the function on the Poincaré disk D, defined in geodesics radial coordinates as

$$u_\omega(t,\theta) := U(g_t r_\theta \omega)\,, \quad \text{for all } (t,\theta) \in D\,.$$

Lemma 4.1. *The hyperbolic gradient (in the radial and tangential directions) and Laplacian Δ of the function u_ω are given (in the weak sense) by the following formulas:*

$$\nabla u_\omega(t,\theta) = 2\,(XU, YU + \coth(2t)\Theta U)\,(g_t r_\theta\omega)$$
$$\Delta u_\omega(t,\theta) = 4\left(X^2 + Y^2 + \coth^2(2t)\Theta^2 + 2\coth(2t)Y\Theta\right)U(g_t r_\theta\omega)\,.$$

Proof. By definition we have

$$\frac{\partial u_\omega}{\partial t}(t,\theta) = 2(XU)(g_t r_\theta\omega)\,.$$

The computation of the angular derivative is based on the formula

$$\begin{aligned}\exp(\theta\Theta)\exp(2tX) &= \exp(2tX)\exp(-2tX)\exp(\theta\Theta)\exp(2tX)\\ &= \exp(2tX)\mathrm{Ad}_{\exp(-2tX)}(\exp(\theta\Theta))\\ &= \exp(2tX)\exp(e^{\mathrm{ad}_{-2tX}}(\theta\Theta))\\ &= \exp(2tX)\exp(\theta(\cosh(2t)\Theta+\sinh(2t)Y))\,.\end{aligned}$$

The above formula is computed with respect to standard generators $\{X, Y, \Theta\}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ which satisfy the commutation relations

$$[\Theta, X] = Y\,, \quad [\Theta, Y] = -X\,, \quad [X, Y] = -\Theta\,.$$

Under the convention in **[For02]**, the curvature of the Poincaré plane is taken to be -4, which corresponds to the choice of generators $\{2X, 2Y, \Theta\}$.

It follows that

$$\frac{\partial u_\omega}{\partial\theta}(t,\theta) = (\cosh(2t)\Theta + \sinh(2t)Y)U(g_t r_\theta \omega)\,.$$

We can now compute the hyperbolic gradient and Laplacian. We have

$$\begin{aligned}\nabla u_\omega(t,\theta) &= \left(\frac{\partial u_\omega}{\partial t}(t,\theta), \frac{2}{\sinh(2t)}\frac{\partial u_\omega}{\partial\theta}(t,\theta)\right)\\ &= 2(XU, (Y+\coth(2t)\Theta)U)(g_r r_\theta\omega)\,.\end{aligned}$$

and, by the commutation relation $[\Theta, Y] = -X$, we also have

$$\begin{aligned}\Delta u_\omega(t,\theta) &= \left(\frac{\partial^2}{\partial t^2} + 2\coth(2t)\frac{\partial}{\partial t} + \frac{4}{\sinh^2(2t)}\frac{\partial^2}{\partial\theta^2}\right)u_\omega(t,\theta)\\ &= (4X^2 + 4\coth(2t)X + \frac{4}{\sinh^2(2t)}(\cosh(2t)\Theta+\sinh(2t)Y)^2)U(g_t r_\theta\omega)\\ &= (4(X^2+Y^2+\coth^2(2t)\Theta^2 + 2\coth(2t)Y\Theta))U(g_t r_\theta\omega)\,.\end{aligned}$$

The computation is completed.

□

For every $t > 0$, let then

$$\begin{aligned}\mathcal{D}_t &:= (X, Y+\coth(2t)\Theta)\\ \mathcal{L}_t &:= -\left(X^2+Y^2+\coth^2(2t)\Theta^2\right) - 2\coth(2t)Y\Theta\,.\end{aligned}$$

Let $W^{2,2}(X,\nu)$ denote the (foliated) Sobolev space of functions which belong to $L^2(X,\nu)$ together with all their derivatives up to second order, in all directions tangent to $\mathrm{SL}(2,\mathbb{R})$ orbits:

$$f \in W^{2,2}(X,\nu) \Longleftrightarrow f,\ \mathcal{V}f,\ \mathcal{V}\mathcal{W}f \in L^2(X,\nu)\,, \qquad \text{for all } \mathcal{V},\mathcal{W}\in\mathfrak{sl}(2,\mathbb{R})\,.$$

By Corollary A.2, for all $t > 0$ the operators $\mathcal{L}_t$ are closed on $W^{2,2}(X,\nu)$, so that, for every $f \in W^{2,2}(X,\nu)$, we have in $L^2(X,\nu)$ the limit

$$\lim_{t\to+\infty}\mathcal{L}_t(f) = \mathcal{L}(f) := -\left(X^2+Y^2+\Theta^2+2Y\Theta\right)f\,,$$

and the convergence is exponential in the sense that, there exists a constant $C > 0$ such that, for all $t > 0$,

$$\|\mathcal{L}_t(f) - \mathcal{L}(f)\|_{L^2(X,\nu)} \leq C\|f\|_{W^{2,2}(X,\nu)} \cdot e^{-t}\,.$$

Similarly, let $W^{1,2}(X, \nu)$, denote the space of functions $f \in L^2(X, \nu)$ such that $\mathcal{V}(f) \in L^2(X, \nu)$ for all $\mathcal{V} \in \mathfrak{sl}(2, \mathbb{R})$. We have that, for $f \in W^{1,2}(X, \nu)$,

$$\lim_{t\to+\infty} \mathcal{D}_t(f) = \mathcal{D}(f) := (Xf, (Y + \Theta)f)\,,$$

and the convergence is exponential.

We note that, since $[\Theta, Y] = -X$, the operator $\mathcal{L}$ can be written as

$$\mathcal{L} = -(X^2 + (Y + \Theta)^2 + X)$$

and $Y + \Theta$ is the generator of the stable horocycle flow. The operator $\mathcal{L}$ is the generator of the foliated Brownian motion on the stable foliation.

Let $E^+_k(\omega)$ denotes the unstable k-dimensional Oseledets subspace at $\omega \in X$ of the A-action on the real Hodge vector bundle (with respect to the canonical $\mathrm{SL}(2, \mathbb{R})$-invariant measure on X). Since the function Φ_k is everywhere bounded, the function $\Phi_k \circ E^+_k$ belongs to the space $L^2(X, \nu)$.

It follows by Corollary A.2 that the equation

$$\mathcal{L}U(\omega) = \frac{1}{2}\left(\Phi_k(\omega, E^+_k(\omega)) - \lambda_{(k)}\right), \qquad \text{for } \omega \in X\,, \tag{4.1.5}$$

has a solution $U^{(k)} \in W^{2,2}(X, \nu)$, the space of functions with all $\mathfrak{sl}(2, \mathbb{R})$-derivatives up to second order in $L^2(X, \nu)$.

Remark 4.2. Recall that the trajectory ρ was lifted to $\mathrm{SL}(2, \mathbb{R})$, and was moreover defined in the introduction by taking the outward radial unit tangent vector at all points. We denote the lifted path by $\bar\rho$. As consequence the unstable subspace $E^+_k(\bar\rho_s)$ is defined at almost all $s \in \mathbb{R}^+$ with probability one as the unstable Oseledets subspace at the radial outward unit tangent vector $\bar\rho_s \in X$ at the point $\rho_s \in D = \mathrm{SO}(2, \mathbb{R})\backslash\mathrm{SL}(2, \mathbb{R})\omega$.

Let $W^{2,\infty}(\mathbb{D})$ denote the Sobolev space of essentially bounded functions on the unit disc $\mathbb{D}$ with essentially bounded weak derivatives up to order 2. The function $u(z) := u^{(k)}_\omega(z)$ belongs to the space $W^{2,\infty}(\mathbb{D})$, and it is not necessarily C^2 along $\mathrm{SL}(2, \mathbb{R})$ orbits. However, a version of Ito's formula for weakly differentiable functions, known as Ito-Krylov's formula (see for instance [**Aeb96, FP00, Kry10**]) applies.

For all $s > 0$, let $\rho_s = (t(s), \theta(s))$ in geodesic polar coordinates. By Ito-Krylov's formula we get,

$$\frac{1}{\sqrt{T}}(u(\rho_T) - u(\rho_0)) = \frac{1}{\sqrt{T}} \int_0^T \nabla_{L_\omega} u(\rho_s) \cdot (dW_s^{(1)}, dW_s^{(2)}) \tag{4.1.6}$$

$$+ \frac{1}{2\sqrt{T}} \int_0^T \Delta_{L_\omega} u(\rho_s) ds \tag{4.1.7}$$

$$= \frac{1}{\sqrt{T}} \int_0^T \nabla_{L_\omega} u(\rho_s) \cdot (dW_s^{(1)}, dW_s^{(2)}) \tag{4.1.8}$$

$$+ \frac{2}{\sqrt{T}} \int_0^T \mathcal{L}_{t(s)} U(\bar{\rho}_s) ds \tag{4.1.9}$$

So we have that

$$\frac{1}{2\sqrt{T}} \left(\int_0^T (\Delta_{L_\omega} \sigma_k(\rho_s, \mathbf{v}) - 2\lambda_{(k)}) ds \right) \tag{4.1.10}$$

$$= \frac{1}{\sqrt{T}} \left(\int_0^T \left(\Phi_k(\rho_s, \mathbf{v}) - \Phi_k(E_k^+(\bar{\rho}_s)) \right) ds \right) \tag{4.1.11}$$

$$+ \frac{2}{\sqrt{T}} \left(\int_0^T (\mathcal{L}U)(\bar{\rho}_s) ds \right) \tag{4.1.12}$$

$$= \frac{1}{\sqrt{T}}(u(\rho_T) - u(\rho_0)) \tag{4.1.13}$$

$$- \frac{1}{\sqrt{T}} \int_0^T \nabla_{L_\omega} u(\rho_s) \cdot (dW_s^{(1)}, dW_s^{(2)}) \tag{4.1.14}$$

$$+ \frac{1}{\sqrt{T}} \left(\int_0^T \left(\Phi_k(\rho_s, \mathbf{v}) - \Phi_k(E_k^+(\bar{\rho}_s)) \right) ds \right) \tag{4.1.15}$$

$$+ \frac{2}{\sqrt{T}} \int_0^T (\mathcal{L}U - \mathcal{L}_{t(s)} U)(\bar{\rho}_s) ds \,. \tag{4.1.16}$$

Define

$$M_T = \int_0^T \nabla_{L_\omega} (\sigma_k(\rho_s, \mathbf{v}) - u(\rho_s)) \cdot (dW_s^{(1)}, dW_s^{(2)}) \tag{4.1.17}$$

We then have

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_T, \mathbf{v}) - T \sum_{i=1}^k \lambda_i) = \frac{1}{\sqrt{T}}(u(\rho_T) - u(\rho_0) + \sigma_k(\rho_0, \mathbf{v})) \tag{4.1.18}$$

$$+ \frac{1}{\sqrt{T}} M_T + \frac{1}{\sqrt{T}} \int_0^T (\Phi_k(\rho_s, \mathbf{v}) - \Phi_k(E_k^+(\bar{\rho}_s))) ds \tag{4.1.19}$$

$$+ \frac{2}{\sqrt{T}} \int_0^T (\mathcal{L}U - \mathcal{L}_{t(s)} U)(\bar{\rho}_s) ds \,. \tag{4.1.20}$$

We observe that, for all $s > 0$, since $(\rho_s, \mathbf{v}) = g_{t(s)} r_{\theta(s)} \mathbf{v}$ and by definition $\bar{\rho}_s = g_{t(s)} r_{\theta(s)} \omega$, the subspace $E_k^+(\bar{\rho}_s)$ is the stable space for the cocycle, defined by parallel transport, over the Brownian motion ρ, in the sense that, for ν-almost all $\omega \in X$ with probability one, by the Oseledets theorem

$$\operatorname{dist}\big((\rho_s, \mathbf{v}), E_k^+(\bar\rho_s)\big) \to 0$$

exponentially fast (as $s \to +\infty$), with respect to the projective distance on the bundle $\mathbb{P}(\mathbf{H}^{(k)})$, hence, taking into account that the function Φ_k is Lipschitz,

$$\frac{1}{T}\mathbb{E}_{\hat\nu_{\mathbb{P}}}\left[\left(\int_0^T \big(\Phi_k(\rho_s, \mathbf{v}) - \Phi_k(E_k^+(\bar\rho_s))\big)\, ds\right)^2\right] \to 0\,.$$

In addition, since $\Theta^2 U$ and $Y\Theta U \in L^2(X)$ by Corollary A.2, we have

$$\frac{1}{T}\mathbb{E}_{\nu}\left[\left(\int_0^T (\mathcal{L}U - \mathcal{L}_{t(s)}U)(\bar\rho_s) ds\right)^2\right] \to 0\,.$$

Next, we study the quadratic variation $\langle M_T, M_T\rangle_{\hat\nu_{\mathbb{P}}}$. Recalling that the covariance of two Ito integrals with respect to independent Brownian motions is zero, we have:

$$\langle M_T, M_T\rangle_{\hat\nu_{\mathbb{P}}} = \mathbb{E}_{\hat\nu_{\mathbb{P}}}\left[\left(\int_0^T (\nabla_{L_\omega}\sigma_k(\rho_s, \mathbf{v}) - \nabla_{L_\omega}u(\rho_s))\cdot (dW_s^{(1)}, dW_s^{(2)})\right)^2\right] \tag{4.1.21}$$

$$= \mathbb{E}_{\hat\nu_{\mathbb{P}}}\left[\left(\int_0^T (\frac{\partial \sigma_k}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s))dW_s^{(1)}\right)^2\right] \tag{4.1.22}$$

$$+ \mathbb{E}_{\hat\nu_{\mathbb{P}}}\left[\left(\int_0^T \frac{2}{\sinh(2t(s))}(\frac{\partial \sigma_k}{\partial \theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial \theta}(\rho_s))dW_s^{(2)}\right)^2\right] \tag{4.1.23}$$

Applying Ito's isometry [**FLJ12**, Lemma VI.4.3] on the expectation of the square of the Ito integrals on the RHS yields

$$\langle M_T, M_T\rangle_{\hat\nu_{\mathbb{P}}} = \mathbb{E}_{\hat\nu_{\mathbb{P}}}\left[\int_0^T \left((\frac{\partial \sigma_k}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s))\right)^2 ds\right] \tag{4.1.24}$$

$$+ \mathbb{E}_{\hat\nu_{\mathbb{P}}}\left[\int_0^T \left(\frac{2}{\sinh(2t(s))}(\frac{\partial \sigma_k}{\partial \theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial \theta}(\rho_s))\right)^2 ds\right] \tag{4.1.25}$$

$$= \mathbb{E}_{\hat\nu_{\mathbb{P}}}\left[\int_0^T |\nabla_{L_\omega}\sigma_k(\rho_s, \mathbf{v}) - \nabla_{L_\omega}u(\rho_s)|^2 ds\right] \tag{4.1.26}$$

$$= \mathbb{E}_{\hat\nu_{\mathbb{P}}}\left[\int_0^T |\nabla_{L_\omega}\sigma_k(\rho_s, \mathbf{v}) - 2\mathcal{D}_{t(s)}U(\rho_s)|^2 ds\right]. \tag{4.1.27}$$

Observe that $XU, YU, \Theta U \in L^2(X,\nu)$ by Corollary A.2. Hence in particular

$$\|\mathcal{D}_t U - \mathcal{D}U\|_{L^2(X)} \to 0$$

exponentially fast. Therefore, by Oseledets' theorem, Fubini's theorem, and the dominated convergence theorem, we have the convergence with respect to the measure $\hat{\nu}$ on $\mathbb{P}(\mathbf{H}^{(k)})$:

$$V_{\rho_\infty}^{(k)} := \lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T |(\nabla_{L_\omega}\sigma_k)(\rho_s,\mathbf{v}) - 2\mathcal{D}_{t(s)}U(\rho_s)|^2 ds\right] \tag{4.1.28}$$

$$= \int_{\mathbb{P}(\mathbf{H}^{(k)})} |\Psi_k(\mathbf{v}) - 2\mathcal{D}U(\omega)|^2 d\hat{\nu} \tag{4.1.29}$$

$$= \int_X |\Psi_k(E_k^+(\omega)) - 2\mathcal{D}U(\omega))|^2 d\nu\,. \tag{4.1.30}$$

See also [**For02**, Corollary 5.5]. The above formula, together with [**FLJ12**, Lemma VIII.7.4], implies that the random variables $M_T/\sqrt{T}$, hence the random variables $(\sigma(\rho_T,\mathbf{v}) - \lambda_{(k)}T)/\sqrt{T}$, converge in distribution to a centered Gaussian distribution of variance $V_{\rho_\infty}^{(k)}$. □

4.2. **Distributional Convergence in Theorem 2.1.** Observe that $t(s) = d_{\mathbb{D}}(0,\rho_s)$, and that it is rotationally invariant. We will need the following useful lemma:

Lemma 4.3. [**FLJ12**, Lemma VII.7.2.1] *For all $\omega \in X$, there exists an $\mathbb{P}_\omega$-almost everywhere converging process η_s such that $t(s) = W_s^{(1)} + s + \eta_s$.*

Proof. It is a classical fact that $t(s) \to \infty$ $\mathbb{P}_\omega$-almost everywhere. This implies that $\lim_{s\to\infty}\coth(2t(s)) = 1$ almost everywhere. Setting $\eta_s := t(s) - W_s^{(1)} - s$, so that, together with 3.8.3, we get

$$\eta_s = \int_0^s (\coth(2t(s)) - 1)ds = \int_0^s \frac{2ds}{e^{4t(s)} - 1},$$

which converges almost everywhere, as desired. □

Next, it will be crucial to stop the radial process before it exits the region bounded by a circle of geodesic radius T, and so for each T, we define the stopping time τ_T as follows

$$\tau_T := \inf\{s > 0 \ : \ T = d_{\mathbb{D}}(0,\rho_s)\} \tag{4.2.1}$$

$$= \inf\{s > 0 \ : \ T = W_s^{(1)} + s + \eta_s\} \tag{4.2.2}$$

where the second equality follows by Lemma 4.3. Next, we will need the following lemma:

Lemma 4.4. *For all $\omega \in X$, we have $\lim_{T\to\infty}\tau_T/T = 1$ $\mathbb{P}_\omega$-almost everywhere. Moreover, we have that as $T \to \infty$, $\tau_t \to \infty$ $\mathbb{P}_\omega$-almost everywhere.*

Proof. Observe that we have $\tau_T = T - W^{(1)}_{\tau_T} - \eta_{\tau_T}$. The lemma then follows immediately from the definition of the stopping time and the law of the iterated logarithm. $\square$

See also [**EFLJ01**, Lemma 4.2] for related and interesting results on this stopping time.

Recall that $\mathbb{P}_\omega$ is the Wiener measure on the space of all Brownian trajectories W_ω starting at the origin (corresponding to the random point ω). Let $\mathbb{P}^\theta_\omega$ be the Wiener measure on the space W^θ_ω corresponding to all paths starting at the origin and conditioned to exit at the point $e^{i\theta}$ in $\partial\mathbb{D}^2$. To relate the conditioned process ρ^θ_s to the unconditioned process ρ_s, we will need the following lemma:

Lemma 4.5.

$$\mathbb{P}_\omega = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}^\theta_\omega \, d\theta \tag{4.2.3}$$

Proof. Recall that W_ω is the space of all hyperbolic Brownian motion trajectories starting at the origin, with $\mathbb{P}_\omega$ the corresponding Wiener measure. There exists a map $\Theta : W_\omega \to \partial\mathbb{D}^2$, defined $\mathbb{P}_\omega$-almost everywhere, such that $\Theta(\rho) = \rho_\infty$, where ρ_∞ is the limit point of ρ on $\partial\mathbb{D}^2$. It is a classical fact that the pushforward measure $\Theta_*(\mathbb{P}_\omega)$ equals Leb, where Leb is the normalized Lebesgue measure on $[0, 2\pi]$. We also recall that the foliated process is in fact defined on $\mathrm{SO}(2,\mathbb{R})\backslash X$ and that $\hat{\nu}$ is $\mathrm{SO}(2,\mathbb{R})$-invariant, and so our disintegration claim follows. $\square$

Remark 4.6. See also [**Fra05**, Lemma 8] for a short potential theoretic proof (using Doob's h-process) of this fact. The approach to proving the central limit theorem in [**Fra05**], with the aid of a stopping time, is what we will essentially follow in the sequel, though in our case the proof here is simpler, in view of the Lipschitz property of the Kontsevich-Zorich cocycle and Ancona's estimate.

Remark 4.7. It is worth repeating and adapting what is written in the introduction in view of the application of the conditioned process in the sequel. The conditioned process is in fact defined on $X^* = \mathrm{SO}(2,\mathbb{R})\backslash X$. Moreover, ρ^θ can be lifted to $\mathrm{SL}(2,\mathbb{R})$, and is moreover defined by taking the outward radial unit tangent vector at all points. We continue to refer to the lifted path as ρ^θ by abuse of notation. Additionally, the space X gives rise to a product space $X^{W^\theta} := X \otimes W^\theta$ whose fiber over each point ω in X is W^θ_ω, and which also supports a measure $\nu_{\mathbb{P}^\theta} := \nu \otimes \mathbb{P}^\theta$, whose conditional measure over a point ω is $\mathbb{P}^\theta_\omega$. We can thus similarly define the product W^θ-Hodge bundle $\mathbb{P}^{W^\theta}(\mathbf{H}^{(k)})$, whose fiber over each point (ω, ρ^θ) in X^{W^θ} is $\mathbf{H}^{(k)}_\omega$. A pair $(\rho^\theta, \mathbf{v}) \in \mathbb{P}^{W^\theta}(\mathbf{H}^{(k)})$ is thus defined to be the lift of the path ρ^θ (starting at ω) to $\mathbb{P}^{W^\theta}(\mathbf{H}^{(k)})$, obtained by parallel transport with respect to the Gauss-Manin connection. This in turn would also give rise to a measure $\hat{\nu}_{\mathbb{P}^\theta} := \hat{\nu} \otimes \mathbb{P}^\theta$ whose conditional measure over a point $\mathbf{v}$ is $\mathbb{P}^\theta_\omega$.

We recall the following fundamental result due to Ancona [**Anc90**] (see also [**Gru98**, Lemma 4.1]):

Theorem 4.8. [**Anc90**, Théorème 7.3] *For all $\omega \in X$, and $\mathbb{P}_\omega$-almost all paths ρ starting at ω, we have that $d_{\mathbb{D}}(\rho_0\rho_\infty, \rho_T) = O(\log T)$ as $T \to \infty$, where $\rho_0\rho_\infty$ is the geodesic ray with $\rho_0 \in \mathbb{D}$ and $\rho_\infty \in \partial\mathbb{D}$.*

Now observe that our aim is to study

$$\Sigma^g(T,[a,b]) := \hat{\nu}\left(\left\{\mathbf{v} \in \mathbb{P}(\mathbf{H}^{(k)}) \;:\; a \leq \frac{1}{\sqrt{T}}(\sigma_k(g_T,\mathbf{v}) - T\lambda_{(k)}) \leq b\right\}\right) \tag{4.2.4}$$

as $T \to \infty$.

Let

$$\Sigma^\rho(T,[a,b]) := \hat{\nu}_{\mathbb{P}}\left(\left\{(\rho,\mathbf{v}) \in \mathbb{P}^W(\mathbf{H}^{(k)}) \;:\; a \leq \frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau_T},\mathbf{v}) - T\lambda_{(k)}) \leq b\right\}\right) \tag{4.2.5}$$

Lemma 4.9. *The quantity*

$$|\Sigma^g(T,[a,b]) - \Sigma^\rho(T,[a,b])| \to 0 \tag{4.2.6}$$

as $T \to \infty$, $\mathbb{P}_\omega$-almost everywhere and for all $\omega \in X$.

Proof. By applying the disintegration in Lemma 4.5, 4.2.5 is also equal to

$$\Sigma^\rho(T,[a,b]) = \mathrm{Leb} \otimes \hat{\nu}_{\mathbb{P}^\theta}\Big(\Big\{(\theta,\rho^\theta,\mathbf{v}) \in [0,2\pi] \otimes \mathbb{P}^{W^\theta}(\mathbf{H}) \;: \tag{4.2.7}$$

$$a \leq \frac{1}{\sqrt{T}}(\sigma_k(\rho^\theta_{\tau_T},\mathbf{v}) - T\lambda_{(k)}) \leq b\Big\}\Big) \tag{4.2.8}$$

$$= \mathrm{Leb} \otimes \hat{\nu}_{\mathbb{P}^\theta}\Big(\Big\{(\theta,\rho^\theta,\mathbf{v}) \in [0,2\pi] \otimes \mathbb{P}^{W^\theta}(\mathbf{H}) \;: \tag{4.2.9}$$

$$a \leq \frac{1}{\sqrt{T}}\Big(\sigma_k(g_Tr_\theta,\mathbf{v}) - T\lambda_{(k)} + \sigma_k(\rho^\theta_{\tau_T},\mathbf{v}) - \sigma_k(g_Tr_\theta,\mathbf{v})\Big) \leq b\Big\}\Big)\,. \tag{4.2.10}$$

Theorem 4.8 applied to τ_T gives that, for all $\omega \in X$, $d_{\mathbb{D}}(g_Tr_\theta \cdot 0, \rho^\theta_{\tau_T}) = O(\log \tau_T)$ $\mathbb{P}^\theta_\omega$-almost everywhere as $T \to \infty$. Together with Lemma 4.4, the lemma now follows by the Lipschitz property of the Kontsevich-Zorich cocycle (by the derivative bound in 3.5.1). □

Therefore, it suffices to study the limiting distribution of the quantity

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau_T},\mathbf{v}) - T\lambda_{(k)}).$$

Observe that we have that for all $\omega \in X$, and $\mathbb{P}_\omega$-almost everywhere, $\tau_t \to \infty$ as $T \to \infty$. By applying the stopping time identity $T = \tau_T + W^{(1)}_{\tau_T} + \eta_{\tau_T}$, a straightforward calculation shows the following equality:

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau_T},\mathbf{v}) - T\lambda_{(k)}) = -\frac{1}{\sqrt{T}}\eta_{\tau_T}\lambda_{(k)} \tag{4.2.11}$$

$$-\frac{1}{\sqrt{T}}W^{(1)}_{\tau_T}\lambda_{(k)} \tag{4.2.12}$$

$$+\frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau_T},\mathbf{v}) - \tau_T\lambda_{(k)}) \tag{4.2.13}$$

So this reduces the proof of the theorem to controlling three terms on the RHS of the previous equality. First, we can observe that 4.2.11 clearly converges to zero $\mathbb{P}_\omega$-almost everywhere by Lemma 4.3. Next, it follows by Lemma 4.4 that

$$\lim_{T\to\infty}\frac{\lambda_{(k)}}{\sqrt{T}}W^{(1)}_{\tau_T} \xrightarrow{d} W^{(1)}_{\lambda^2_{(k)}}$$

and in particular the variance of 4.2.12 is $\lambda^2_{(k)}$. The variance of 4.2.13 converges to $V^{(k)}_{\rho_\infty}$ by Theorem 2.5, together with the simple observation that, since by Lemma 4.4 $\tau_T/T \to 1$ and M_T is a (deterministic) Ito process (integral) we have

$$\lim_{T\to+\infty}\frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau_T},\mathbf{v}) - \tau_T\lambda_{(k)}) \tag{4.2.14}$$

$$= \lim_{T\to+\infty}\frac{1}{\sqrt{T}}M_{\tau_T} = \lim_{T\to+\infty}M_{\tau_T/T} = M_1\,, \tag{4.2.15}$$

and similarly

$$\lim_{T\to+\infty}\frac{1}{\sqrt{T}}(\sigma_k(\rho_T,\mathbf{v}) - T\lambda_{(k)}) \tag{4.2.16}$$

$$= \lim_{T\to+\infty}\frac{1}{\sqrt{T}}M_T = M_1\,. \tag{4.2.17}$$

Remark 4.10. In fact, by the above argument it follows also that $\frac{1}{\sqrt{\tau_T}}M_{\tau_T}$ converges in distribution to a centered Gaussian random variable with variance $V^{(k)}_{\rho_\infty}$.

The following lemma concerns the covariance of the terms 4.2.12 and 4.2.13, and shows that it converges almost everywhere:

Lemma 4.11. *$Cov_{\hat{\nu}_{\mathbb{P}}}\left(\frac{1}{\sqrt{T}}M_{\tau_T}, -\frac{\lambda_{(k)}}{\sqrt{T}}W^{(1)}_{\tau_T}\right) \to -\lambda^2_{(k)}$*

Proof. It follows by Eqs 3.6.1 and 4.1.5 that we have

$$\begin{aligned}
\Delta_{L_\omega}\sigma_k(z,\mathbf{v}) - \Delta_{L_\omega}u(z) &= 2\left(\Phi_k(g_t r_\theta\omega,\mathbf{v}) - 2\mathcal{L}_t U(g_t r_\theta\omega)\right) \\
&= 2\left(\Phi_k(g_t r_\theta\omega,\mathbf{v}) - 2\mathcal{L}U(g_t r_\theta\omega)\right) + 4(\mathcal{L}U - \mathcal{L}_t U)(g_t r_\theta\omega) \\
&= 2\lambda_{(k)} + 2(\Phi_k(g_t r_\theta\omega,\mathbf{v}) - \Phi_k(g_t r_\theta\omega, E^+_k(g_t r_\theta\omega))) \\
&\quad + 4(\mathcal{L}U - \mathcal{L}_t U)(g_t r_\theta\omega))\,.
\end{aligned}$$

Oseledets theorem together with smoothness properties of the function U, namely that $\Theta^2 U, Y\Theta U \in L^2(X,\nu)$, then imply that, for almost all $\omega \in X$ and for almost all $\theta \in [0,2\pi)$, we have,

$$\Phi_k(g_t r_\theta \omega, E_k^+(g_t r_\theta \omega))) + 4(\mathcal{L}U - \mathcal{L}_t U)(g_t r_\theta \omega)) \to 0\,,$$

hence, by [**For02**, Lemma 3.1], we conclude that

$$\frac{\partial}{\partial t}\frac{1}{2\pi}\int_0^{2\pi}(\sigma_k(z,\mathbf{v}) - u(z))d\theta \tag{4.2.18}$$

$$= \frac{1}{\sinh(2t)}\int_0^t \frac{1}{2\pi}\int_0^{2\pi}(\Delta_{L_\omega}\sigma_k - \Delta_{L_\omega}u)d\theta \sinh(2r)dr \tag{4.2.19}$$

$$= \lambda_{(k)}\frac{\cosh(2t)-1}{\sinh(2t)} + o(1) = \lambda_{(k)}\tanh(t) + o(1)\,. \tag{4.2.20}$$

We are now ready to calculate the covariance. We have

$$\mathrm{Cov}_{\hat{\nu}_{\mathbb{P}}}\left(\frac{1}{\sqrt{T}}M_{\tau_T}, -\frac{\lambda_{(k)}}{\sqrt{T}}W^{(1)}_{\tau_T}\right) \tag{4.2.21}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda_{(k)}}{T}\int_0^{\tau_T}(\nabla_{L_\omega}\sigma(\rho_s,\mathbf{v}) - \nabla_{L_\omega}u(\rho_s))\cdot(dW_s^{(1)}, dW_s^{(2)})\int_0^{\tau_T}dW_s^{(1)}\right] \tag{4.2.22}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda_{(k)}}{T}\int_0^{\tau_T}(\frac{\partial\sigma}{\partial t}(\rho_s,\mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s))dW_s^{(1)}\int_0^{\tau_T}dW_s^{(1)}\right] \tag{4.2.23}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda_{(k)}}{T}\int_0^{\tau_T}(\frac{\partial\sigma}{\partial t}(\rho_s,\mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s))ds\right] \tag{4.2.24}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda_{(k)}}{T}\int_0^{T}(\frac{\partial\sigma}{\partial t}(\rho_s,\mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s))ds\right] + o(1) \tag{4.2.25}$$

$$= -\frac{\lambda_{(k)}^2}{T}\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T \tanh(t(s))ds\right] + o(1) \tag{4.2.26}$$

$$\to -\lambda_{(k)}^2\,, \tag{4.2.27}$$

where 4.2.23 follows by the independence of $W_s^{(1)}$ and $W_s^{(2)}$, and where 4.2.24 follows by an application of Ito's inner product (a more general case of Ito's isometry, which follows by applying the polarization identity), which also holds for our stopping time – in fact, Ito's isometry holds for stochastic integrals with infinite time horizon, and so it also follows for our defined stopping time (see also [**FLJ12**, Lemma VI.4.3]). We also note that 4.2.25 holds thanks to Lemma 4.4 and the identity within its proof. Finally, 4.2.26 holds thanks to 4.2.20, together with the rotational invariance of the hyperbolic heat kernel. □

To conclude the proof of Theorem 2.1, and by writing $W^{(1)}_{\tau_T} = \int_0^{\tau_T} dW^{(1)}_s$, we have

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau_T}, \mathbf{v}) - T\lambda_{(k)}) \tag{4.2.28}$$
$$= -\frac{1}{\sqrt{T}} W^{(1)}_{\tau_T}\lambda_{(k)} + \frac{1}{\sqrt{\tau_T}} M_{\tau_T} + o(1) \tag{4.2.29}$$
$$= \frac{1}{\sqrt{T}}\int_0^{\tau_T}(-\lambda_{(k)} + \frac{\partial\sigma}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s))dW^{(1)}_s \tag{4.2.30}$$
$$+ \frac{1}{\sqrt{T}}\int_0^{\tau_T} \frac{2}{\sinh(2t(s))}(\frac{\partial\sigma}{\partial\theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial\theta}(\rho_s))dW^{(2)}_s + o(1) \tag{4.2.31}$$

Since $\tau_T = T - W^{(1)}_{\tau_T} - \eta_{\tau_T}$, we have

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau_T}, \mathbf{v}) - T\lambda_{(k)}) + o(1) \tag{4.2.32}$$
$$= \frac{1}{\sqrt{T}}\int_0^{T}(-\lambda_{(k)} + \frac{\partial\sigma}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s))dW^{(1)}_s \tag{4.2.33}$$
$$+ \frac{1}{\sqrt{T}}\int_0^{T} \frac{2}{\sinh(2t(s))}(\frac{\partial\sigma}{\partial\theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial\theta}(\rho_s))dW^{(2)}_s \tag{4.2.34}$$
$$- \frac{1}{\sqrt{T}}\int_{T-W^{(1)}_{\tau_T}-\eta_{\tau_T}}^{T}(-\lambda_{(k)} + \frac{\partial\sigma}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s))dW^{(1)}_s \tag{4.2.35}$$
$$- \frac{1}{\sqrt{T}}\int_{T-W^{(1)}_{\tau_T}-\eta_{\tau_T}}^{T} \frac{2}{\sinh(2t(s))}(\frac{\partial\sigma}{\partial\theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial\theta}(\rho_s))dW^{(2)}_s \tag{4.2.36}$$

We remark that the stochastic integrals 4.2.35 and 4.2.35 converge to 0 in probability. Indeed, since the Brownian motion $W^{(1)}$ has mean zero and η_t is convergent, we have that $(W^{(1)}_{\tau_T} + \eta_{\tau_T})/T \to 0$, hence for any square integrable function f it follows that

$$\frac{1}{\sqrt{T}}\int_{T-W^{(1)}_{\tau_T}-\eta_{\tau_T}}^{T} f(\rho_s)dW^{(1)}_s \tag{4.2.37}$$
$$= \int_{T-W^{(1)}_{\tau_T}-\eta_{\tau_T}}^{T} f(\rho_s)dW^{(1)}_{s/T} \tag{4.2.38}$$
$$= \int_{1-(W^{(1)}_{\tau_T}+\eta_{\tau_T})/T}^{1} f(\rho_{Ts})dW^{(1)}_s \to 0\,. \tag{4.2.39}$$

In addition, it follows by [**FLJ12**, Lemma VIII.7.4] that the sum of the normalized stochastic integrals 4.2.33 and 4.2.34 converges to a centered Gaussian distribution. This, together with convergence of the asymptotic covariance in Lemma 4.11, completes the proof, and in particular we have that the asymptotic variance $V^{(k)}_{g_\infty}$ is

$$V^{(k)}_{g_\infty} = V^{(k)}_{\rho_\infty} + \lambda^2_{(k)} + 2\lim_{T\to\infty} \mathrm{Cov}\left(\frac{2}{\sqrt{T}} M_{\tau_T}, -\frac{\lambda_{(k)}\sqrt{2}}{\sqrt{T}} W^{(1)}_{\tau_T}\right) \tag{4.2.40}$$

$$= V^{(k)}_{\rho_\infty} + \lambda^2_{(k)} - 2\lambda^2_{(k)} = V^{(k)}_{\rho_\infty} - \lambda^2_{(k)}\,. \tag{4.2.41}$$

□

5. Positivity of the variance

5.1. **Random cocycle.** Recall that 4.2.41 says that $V^{(k)}_{g_\infty} = V^{(k)}_{\rho_\infty} - \lambda^2_{(k)}$, and so we also have the following important corollary:

Corollary 5.1. *If $\lambda_k > \lambda_{k+1}$, then $V^{(k)}_{\rho_\infty} > 0$.*

Proof. Since, by construction, $V^{(k)}_{g_\infty} \geq 0$, and we have that $V^{(k)}_{\rho_\infty} \geq \lambda^2_{(k)} > 0$, and it is clear that, since $\lambda_{(k)} = \sum_{i=1}^k \lambda_i$, we have $\lambda^2_{(k)} \geq \lambda_1^2 > 0$. □

5.2. **Deterministic cocycle.** While 4.2.41 ensures convergence of the asymptotic variance for the deterministic cocycle, it is not clear to us how it can be leveraged to deduce its positivity. Instead, we approach the positivity of the variance for the deterministic cocycle directly, in the spirit of the potential theoretic approach in **[For02]**. We first observe that a direct expression of the converging asymptotic variance for the deterministic cocycle is

$$V^{(k)}_{g_\infty} = \lim_{T\to\infty} \frac{1}{T}\int_{\mathbb{P}(\mathbf{H})} \left[\sigma_k(g_T, \mathbf{v}) - \lambda_{(k)}T\right]^2 d\hat{\nu} \tag{5.2.1}$$

The existence and the regularity of the solution U of the Poisson equation 4.1.5 will be again crucial for our approach towards the positivity of $V^{(k)}_{g_\infty}$ (see also the proof of Lemma 4.11).

Let $F_k(g_T r_\theta, \mathbf{v}) := \sigma_k(g_T r_\theta, \mathbf{v}) - \lambda_{(k)}T$. In fact, we will study an auxiliary random variable $F_k - U$, and use it at the end to deduce the positivity of the asymptotic variance $V^{(k)}_{g_\infty}$.

Let Ψ_k the vector valued function defined in formula (3.6.2):

$$\Psi_k(\omega, \mathbf{v}) = \mathrm{tr}\left(B^{(k)}_\omega(\mathbf{v})\right)\,, \qquad \text{for all } (\omega, \mathbf{v}) \in \mathbb{P}(\mathbf{H}^{(k)})\,,.$$

We prove below the following condition for the vanishing of the deterministic variance.

Lemma 5.2. *The variance $V^{(k)}_{g_\infty}$ of the deterministic cocycle (see formula (5.2.1)) vanishes if and only if*

$$\Psi_k \circ E^+_k - (\lambda_{(k)}, 0) - 2\mathcal{D}U = 0 \qquad \nu\text{-almost everywhere}.$$

Proof. We first prove that the normalized asymptotic variance of the random variable F_k coincides with that of $F_k - U$.

It follows by an immediate application of [**For02**, Lemma 3.1] that, for any smooth function F and any function $u \in W^{2,\infty}$ on the Poincaré disk, with respect to hyperbolic geodesic polar coordinates $z = (t, \theta)$, we have the formula

$$\frac{1}{2\pi}\frac{\partial}{\partial t}\int_0^{2\pi}(F-u)^2(t,\theta)d\theta = \frac{1}{2}\tanh(t)\frac{1}{|D_t|}\int_{D_t}\Delta_{L_\omega}((F-u)^2)\omega_P \tag{5.2.2}$$

$$= \tanh(t)\frac{1}{|D_t|}\int_{D_t}(F-u)\Delta_{L_\omega}(F-u)\omega_P \tag{5.2.3}$$

$$+ \tanh(t)\frac{1}{|D_t|}\int_{D_t}|\nabla_{L_\omega}(F-u)|^2\omega_P \tag{5.2.4}$$

where $|D_t|$ is the hyperbolic area element of the disk D_t of geodesic radius $t > 0$ that is centered at the origin, and ω_P is the hyperbolic area on the Poincaré disk.

By applying the above formula to $F(t,\theta) = F_k(g_t(r_\theta(\omega), \mathbf{v})$ and $u(t,\theta) = U(g_t r_\theta(\omega))$ and by integrating over $\mathbb{P}(\mathbf{H^{(k)}})$ with respect to the measure $\hat{\nu}$ (defined as the $\mathrm{SO}(2,\mathbb{R})$-invariant Haar measures on the fibers), we have

$$\int_{\mathbb{P}(\mathbf{H^{(k)}})}\frac{\partial}{\partial t}(F_k-u)^2 d\hat{\nu} \tag{5.2.5}$$

$$= \int_{\mathbb{P}(\mathbf{H}^{(k)})}\frac{\tanh(t)}{\sinh^2(t)}\int_0^t (F_k-u)\Delta_{L_\omega}(F_k-u)d(\sinh^2\tau)d\hat{\nu} \tag{5.2.6}$$

$$+ \int_{\mathbb{P}(\mathbf{H}^{(k)})}\frac{\tanh(t)}{\sinh^2(t)}\int_0^t |\nabla_{L_\omega}(F_k-u)|^2 d(\sinh^2\tau)d\hat{\nu} \tag{5.2.7}$$

By integrating over $[0,T]$ with respect to dt, we have

$$\frac{1}{T}\left[\int_{\mathbb{P}(\mathbf{H}^{(k)})}[(F_k-u)^2(g_T,\mathbf{v}) - (F_k-u)^2(g_0,\mathbf{v})]d\hat{\nu}\right] \tag{5.2.8}$$

$$= \frac{1}{T}\int_0^T\int_{\mathbb{P}(\mathbf{H}^{(k)})}\frac{\tanh(t)}{\sinh^2(t)}\int_0^t (F_k-u)\Delta_{L_\omega}(F_k-u)d(\sinh^2\tau)d\hat{\nu}dt \tag{5.2.9}$$

$$+ \frac{1}{T}\int_0^T\int_{\mathbb{P}(\mathbf{H}^{(k)})}\frac{\tanh(t)}{\sinh^2(t)}\int_0^t |\nabla_{L_\omega}(F_k-u)|^2 d(\sinh^2\tau)d\hat{\nu}dt \tag{5.2.10}$$

By Eq. 4.1.5, we observe that

$$\Delta_{L_\omega}(F_k - u)(t,\theta) \tag{5.2.11}$$
$$= \Delta_{L_\omega}(\sigma_k(g_t r_\theta \omega, \mathbf{v})) - \Delta_{L_\omega}(\lambda_{(k)} t) - 4\mathcal{L}_t U(g_t r_\theta \omega) \tag{5.2.12}$$
$$= 2\left(\Phi_k(g_t r_\theta \omega, \mathbf{v}) - \Phi_k(g_t r_\theta \omega, E_k^+(g_t r_\theta \omega)\right) \tag{5.2.13}$$
$$+ 2\lambda_{(k)}(1 - \coth(2t)) + 4(\mathcal{L}_t - \mathcal{L})U(g_t r_\theta \omega) \to 0 \tag{5.2.14}$$

exponentially as $t \to \infty$ by Oseledets theorem and by the regularity properties of the function U. It follows therefore that 5.2.9 converges to 0 as $t \to \infty$, and we thus have that

$$\lim_{T\to\infty} \frac{1}{T} \int_{\mathbb{P}(\mathbf{H}^{(k)})} (F_k - U)^2(g_T, \mathbf{v}) d\hat{\nu} \tag{5.2.15}$$
$$= \int_{\mathbb{P}(\mathbf{H}^{(k)})} |\Psi_k(E_k^+(\omega)) - (\lambda_{(k)}, 0) - 2\mathcal{D}U(\omega)|^2 d\hat{\nu}\,. \tag{5.2.16}$$

Remark 5.3. The above steps follow closely the outline of the proof of [**FMZ12**, Theorem 1], which we refer to for more details.

We have therefore shown that the normalized asymptotic variance of $F_k - U$ is strictly positive if the function $\Psi_k \circ E_k^+ - \lambda_{(k)} - \mathcal{D}U$ is not identically zero. The final claim in the argument is that the asymptotic variance of F_k is no smaller than that of $F_k - U$, and this follows by an immediate application of the triangle inequality, as follows

$$\left[\frac{1}{T}\int_{\mathbb{P}(\mathbf{H}^{(k)})} (F_k - U)^2(g_T, \mathbf{v}) d\hat{\nu}\right]^{1/2} \leq \left[\frac{1}{T}\int_{\mathbb{P}(\mathbf{H}^{(k)})} F_k^2(g_T, \mathbf{v}) d\hat{\nu}\right]^{1/2} \tag{5.2.17}$$
$$+ \left[\frac{1}{T}\int_X U^2(g_T \omega) d\nu\right]^{1/2} \tag{5.2.18}$$
$$= \left[\frac{1}{T}\int_{\mathbb{P}(\mathbf{H}^{(k)})} F_k^2(g_T, \mathbf{v}) d\hat{\nu}\right]^{1/2} \tag{5.2.19}$$
$$+ \frac{1}{\sqrt{T}} \|U\|_{L^2(\nu)} \tag{5.2.20}$$

together with the square integrability of U. We have therefore shown that if the asymptotic variance for the deterministic cocycle $V_{g_\infty}^{(k)}$ is equal to zero, then

$$\Psi_k \circ E_k^+(\omega) - (\lambda_{(k)}, 0) - 2\mathcal{D}U(\omega) = 0 \quad \nu\text{-almost everywhere.} \tag{5.2.21}$$

□

We then prove a regularity result for the unstable Oseledets subspace under the assumption of zero variance.

Let $h_t^- = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ denote the stable horocyclic (unipotent) subgroup of the group $\mathrm{SL}(2,\mathbb{R})$. In our notation, the generator of the flow h_t^- is the vector field $H := Y + \Theta$. In fact, we have

$$[X, Y + \Theta] = -\Theta - Y = -(Y + \Theta)$$

and we follow the opposite convention with respect to the paper **[FF03]** and carry out our calculations considering the action of $\mathrm{SL}(2,\mathbb{R})$ on the left (indeed, the generator of the stable horocycle flow is $U = -(Y - \Theta)$ in §2.1 of **[FF03]**).

Lemma 5.4. *Assume the variance $V^{(k)}_{g_\infty} = 0$. Then for ν-almost all $\omega \in X$ the function*

$$\Psi_k(E_k^+(h_s^- \omega)) \,, \quad s \in \mathbb{R} \,,$$

is a Lipschitz function.

Proof. Since by assumption the variance $V^{(k)}_{g_\infty} = 0$ the identity (5.2.21) holds, hence in particular we have

$$XU = \frac{1}{2} \left(\mathrm{Re} \left(\Psi_k \circ E_k^+ \right) - \lambda_{(k)} \right) \,.$$

Since the unstable space E_k^+ of the cocycle is invariant under the Teichmüller flow, it follows that the function $\Psi_k \circ E_k^+$ is for almost all $\omega \in X$ smooth along the Teichmüller orbit, and by a similar argument along the orbit of the unstable Teichmüller horocycle flow. It follows then that the function U is infinitely differentiable, with derivatives uniformly bounded almost everywhere, along the geodesic flow orbit for almost all $\omega \in X$.

By the construction of the function U as a solution of the equation

$$-\left(X^2 + (Y + \Theta)^2 + X \right) U := \mathcal{L} U = \frac{1}{2} (\Phi_k \circ E_\omega^+ - \lambda_{(k)}) \,,$$

hence it follows that, for ν-almost all $\omega \in X$, we have

$$\frac{d^2}{ds^2} U(h_s \omega) = H^2 U(h_s^- \omega)$$

is a bounded function, which in turn implies that

$$\frac{1}{2} \,\mathrm{Im}(\Psi_k \circ E_k^+)(h_s^- \omega) = HU(h_s^- \omega)$$

has bounded uniformly bounded derivative, hence it is a Lipschitz function. For the real part of the function, we argue that

$$\begin{aligned} H\mathrm{Re}(\Psi_k \circ E_k^+) &= HXU = [H, X]U + XHU \\ &= HU + XHU = \frac{1}{2}(I + X)\mathrm{Im}(\Psi_k \circ E_k^+) \end{aligned}$$

which is again a function uniformly bounded almost everywhere, hence the function $\mathrm{Re}(\Psi_k \circ E_k^+)$ is also Lipschitz along almost all horocycle orbits.

□

Let $\mathbf{H}$ be an $\mathrm{SL}(2,\mathbb{R})$-invariant, symplectic subbundle of the Hodge bundle, symplectically orthogonal to the tautological subbundle.

Corollary 5.5. *If the Lyapunov spectrum of the Kontsevich–Zorich cocycle on $\mathbf{H}$ is simple, that is, if $\lambda_1 > \cdots > \lambda_h$, then the deterministic Kontsevich–Zorich cocycle on $\mathbf{H}^{(1)} = \mathbf{H}$ has strictly positive variance, that is, $V^{(1)}_{g_\infty} > 0$.*

Proof. The proof is by contradiction.

Let us assume that $V^{(1)}_{g_\infty} = 0$. By Lemma 5.4 for ν-almost all $\omega \in X$ the function

$$\Psi_1(E_1^+(h_s^-\omega)), \quad s \in \mathbb{R},$$

is a Lipschitz function.

The strategy of the argument, based on the so-called *freezing argument* from **[CF20]**, consists in deriving from the above Lipshitz property the existence of a proper $\mathrm{SL}(2,\mathbb{R})$-invariant subbundle of $\mathbb{P}(\mathbf{H})$, thereby contradicting the strong irreducibility assumption.

Recall that the function Ψ_1 is given by the formula (see formula (3.6.2))

$$\Psi_1(\omega, \mathbf{v}) = \mathrm{tr}(B^{(1)}_\omega(\mathbf{v})) = B_\omega(\mathbf{v}), \qquad \text{for } (\omega, \mathbf{v}) \in \mathbf{H}.$$

Since for every $\omega \in X$ the matrix B_ω is a complex symmetric matrix, with entries given by a complex quadratic form, it follows that the function Ψ_1 is a polynomial function (with respect to projective coordinates) on every fiber $\mathbf{H}_\omega$. In addition, the function Ψ_1 is non-constant along circle orbits, since

$$\Psi_1(r_\theta\omega, \mathbf{v}) = e^{-2i\theta}\Psi_1(\omega, \mathbf{v}), \qquad \text{for all } \mathbf{v} \in \mathbf{H}_\omega.$$

We define a measurable subbundle of the bundle $\mathbf{H}$ as follows. Let $\mathcal{K} \subset X$ denote a compact subset. For every $\omega \in X$ Birkhoff regular for the Teichmüller geodesic flow and Oseledets regular for the Kontsevich–Zorich cocycle, and for every backward return time $-t < 0$ of the Teichmüller geodesic flow to the compact set $\mathcal{K} \subset X$, we let

$$\mathcal{W}(g_{-t}\omega) := \Psi_1^{-1}\left\{\Psi_1(E_1^+(g_{-t}\omega))\right\}. \tag{5.2.22}$$

We note that $\mathcal{W}(g_{-t}\omega)$ is a real analytic submanifold of (real) codimension 2 which contains the point $E_1^+(g_{-t}\omega) \subset \mathbb{P}(H_{g_{-t}\omega})$. We then let, for all $\omega \in X$,

$$\mathcal{V}(\omega) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \{g_t\left(\mathcal{W}(g_{-t}\omega)\right) | g_{-t}\omega \in \mathcal{K}\}}.$$

We remark that since $\lambda_1 > 0$, by Oseledets theorem the set $\mathcal{V}(\omega)$ is contained in a finite union of g_t-invariant subspaces.

By definition, for ν-almost all $\omega \in X$, the set $\mathcal{V}(\omega) \subset \mathbb{P}(\mathbf{H}_\omega)$ is closed and non-empty, since it contains $E_1^+(\omega)$. Since E_1^+ is invariant under the Teichmüller geodesic flow $\{g_t\}$, it is straightforward to prove that $\mathcal{V}$ is a (measurable) $\{g_t\}$-invariant subset.

The crucial point of the argument is to prove that $\mathcal{V}$ is invariant under the stable Teichmüller horocycle flow $\{h_s^-\}$.

Let $\mathbf{v} \in g_t\left(\mathcal{W}(g_{-t}\omega)\right)$. By definition, $\Psi_1(g_{-t}(\mathbf{v})) = \Psi_1(E_1^+(g_{-t}(\omega)))$. There exists a constant $C_{\mathcal{K}}$ such that for any fixed $r > 0$ the distance

$$d(g_{-t}\omega, g_{-t}(h_r^-\omega)) = d(g_{-t}\omega, h^-_{e^{-2t}r}(g_{-t}\omega)) \leq C_{\mathcal{K}} r e^{-2t}\,,$$

hence by the Lipschitz property of the function $\Psi_1 \circ E_1^+$ (which holds by Lemma 5.4) we have that there exists a constant $C'_{\mathcal{K}}$ such that

$$\|(\Psi_1 \circ E_1^+)(g_{-t}\omega) - (\Psi_1 \circ E_1^+)(g_{-t}(h_r^-\omega)))\| \leq C'_{\mathcal{K}} r e^{-2t}\,.$$

We also have (with respect to the Hodge metric)

$$d\left(g_{-t}(h_r^-(\mathbf{v})), g_{-t}(\mathbf{v})\right) \leq r e^{-2t}\,,$$

so that we have the estimate

$$\begin{aligned}
&\|(\Psi_1(g_{-t}(h_r^-\mathbf{v})) - \Psi_1(E_1^+(g_t h_r^-\omega))\| \\
&\quad \leq \|\Psi_1(g_{-t}(h_r^-\mathbf{v})) - \Psi_1(g_{-t}(\mathbf{v}))\| + \|\Psi_1(g_{-t}(\mathbf{v})) - \Psi_1(E_1^+(g_{-t}h_r^-\omega))\| \\
&\quad = \|\Psi_1(g_{-t}(h_r^-\mathbf{v})) - \Psi_1(g_{-t}(\mathbf{v}))\| + \|\Psi_1(E_1^+(g_{-t}(\omega)) - \Psi_1(E_1^+(g_{-t}h_r^-\omega))\| \\
&\quad \leq \|D\Psi_1\|_{\mathcal{K}} d\left(g_{-t}(h_r^-\mathbf{v}), g_{-t}(\mathbf{v})\right) + \|\Psi_1(E_1^+(g_{-t}(\omega)) - \Psi_1(E_1^+(g_{-t}h_r^-\omega))\|\,.
\end{aligned}$$

Thus there exists a constant $C''_{\mathcal{K}} > 0$ such that

$$d\left(g_{-t}(h_r^-\mathbf{v}), \mathcal{W}(g_{-t}h_r^-\omega)\right) \leq C''_{\mathcal{K}} r e^{-2t}\,.$$

Next we claim that there exists $\lambda < 1$ such that the above estimate implies that there exists a constant $C^{(3)}_{\mathcal{K}} > 0$ such that

$$d\left(h_r^-\mathbf{v}, g_t\mathcal{W}(g_{-t}h_r^-\omega)\right) \leq C^{(3)}_{\mathcal{K}} r e^{-2(1-\lambda)t}\,.$$

The above conclusion follows from the fact that there exists $\lambda \in (0,1)$ such that, on the symplectic orthogonal of the tautological bundle, the Lyapunov spectrum is contained in the interval $(-\lambda, \lambda)$. It then follows by Oseledets theorem that for every Birkhoff generic and regular $\omega \in X$ and for every $\mathbf{v}$, $\mathbf{w} \in \mathbb{P}(H_\omega)$,

$$\limsup_{t\to+\infty} \frac{1}{t} \log d(g_t(\mathbf{v}), g_t(\mathbf{w})) \leq 2\lambda\,.$$

We have thus proved that, for every $s > 0$,

$$h_r^-(\mathbf{v}) \in \overline{\bigcup_{t\geq s} \{g_t\left(\mathcal{W}(g_{-t}h_r^-\omega)\right) | g_{-t}h_r^-\omega \in \mathcal{K}\}}$$

hence $h_r^-(\mathbf{v}) \in \mathcal{V}(h_r^-\omega)$, for all $\mathbf{v} \in g_t(\mathcal{W}(g_{-t}\omega)))$. Since $\mathcal{V}(h_r^-\omega)$ is closed, it follows that, for every $r \in \mathbb{R}$,

$$h_r^-\left(\mathcal{V}(\omega)\right) \subset \mathcal{V}(h_r^-\omega)\,,$$

and, since the reverse inclusion can be proved by reversing the time in the horocycle flow, we have proved the invariance of the bundle $\mathcal{V}$ under the unstable Teichmüller horocycle flow.

We claim that by the construction of the bundle $\mathcal{V}$, the unstable bundle $E_1^+ \subset \mathcal{V}$. In fact, by the definition in formula (5.2.22) we have that, for almost all $\omega \in X$ and for all $t \in \mathbb{R}$,

$$E_1^+(g_{-t}\omega) \subset \mathcal{W}(g_{-t}\omega)\,.$$

and since E_1^+ is a $\{g_t\}$-invariant bundle, it follows that, for all $t \geq 0$,

$$E_1^+(\omega) \subset g_t\mathcal{W}(g_{-t}\omega)$$

which implies the claim.

We can then define an $\mathrm{SL}(2,\mathbb{R})$-invariant subbundle as follows. Let $\mathcal{E}$ denote the smallest measurable forward $\{h_r^-\}$-invariant bundle which contains E_1^+. In other terms, for almost all $\omega \in X$, let

$$\mathcal{E}(\omega) := \sum_{r \geq 0} h_r^- E_1^+(h_{-r}^-\omega)\,.$$

We note that, by the above definition, $\mathcal{E} \subset \mathcal{V}$ since $E_1^+ \subset \mathcal{V}$, and the latter bundle is $\{g_t\}$-invariant and $\{h_r^-\}$-invariant.

We then prove that the bundle $\mathcal{E}$ is $\mathrm{SL}(2,\mathbb{R})$-invariant. It is clearly forward $\{h_r^-\}$-invariant by definition. Let us prove that it is $\{g_t\}$-invariant.

By the commutation relation and by the $\{g_t\}$-invariance of the bundle E_1^+, for almost all $\omega \in X$ and for all $t, r \in \mathbb{R}$, we have

$$\begin{aligned} g_t\left(h_r^- E_1^+(h_{-r}^-\omega)\right) &= (h_{e^{-t}r}^- \circ g_t)E_1^+(h_{-r}^-\omega) \\ &= h_{e^{-t}r}^-\left(E_1^+(g_t \circ h_{-r}^-\omega)\right) = h_{e^{-t}r}^- E_1^+(h_{-e^{-t}r}^- \circ g_t\omega) \end{aligned}$$

which immediately implies that, for all $t \in \mathbb{R}$,

$$g_t\mathcal{E}(\omega) = \mathcal{E}(g_t\omega)\,,$$

hence the bundle $\mathcal{E}$ is (forward and backward) $\{g_t\}$-invariant.

Let us then prove that the bundle $\mathcal{E}$ is forward $\{h_s^+\}$-invariant. For the unstable horocycle flow $\{h_s^+\}$ we have the following commutation relations. For every $r, s \in \mathbb{R}$, with $rs \neq -1$, let

$$\rho(r,s) = \frac{r}{1+rs}\,, \quad \sigma(r,s) = s(1+rs)\,, \quad \tau(r,s) = \log(1+rs)\,.$$

We then have the commutation relations:

$$h_s^+ \circ h_r^- = h_\rho^- \circ h_\sigma^+ \circ g_\tau\,.$$

Since E_1^+ is $\{g_t\}$-invariant and $\{h_s^+\}$-invariant, it follows that we have

$$\begin{aligned} h_s^+\left(h_r^- E_1^+(h_{-r}^-\omega)\right) &= h_\rho^- \circ h_\sigma^+ \circ g_\tau\left(E_1^+(h_{-r}^-\omega)\right) \\ &= h_\rho^-\left(E_1^+(h_\sigma^+ \circ g_\tau \circ h_{-r}^-\omega)\right) = h_\rho^-\left(E_1^+(h_{-\rho}^- h_s^+\omega)\right)\,, \end{aligned}$$

which immediately implies that $\mathcal{E}$ is forward $\{h_s^+\}$-invariant. Finally, since $\mathcal{E}$ is $\{g_t\}$-invariant and forward $\{h_s^\pm\}$-invariant, it follows that it is $SL(2,\mathbb{R})$-invariant as claimed.

Finally we remark that by the condition that the Lyapunov spectrum is simple, it follows that $\mathcal{V} \neq \mathbb{P}(\mathbf{H})$, hence $\mathcal{E} \subset \mathcal{V} \neq \mathbb{P}(\mathbf{H})$. In fact, by definition, since for almost all $\omega \in X$ and for all $t \geq 0$, the real analytic sets $\mathcal{W}(g_t\omega)$ have positive codimension equal to 2, by Oseledets theorem, for almost all $\omega \in X$, the subset $\mathcal{V}$ is contained in the union of finitely many proper g_t- invariant sub-bundles of $\mathbb{P}(\mathbf{H})$, given by the Oseledets decomposition, namely all the codimension 2 sums of the one-dimensional Oseledets subspaces, in contradiction with the hypothesis that $\mathbf{H}$ is strongly irreducible. $\square$

Appendix A. Solving Poisson's equation

The purpose of this section is to derive from the spectral gap of the foliated Laplacian due to Avila-Gouëzel-Yoccoz [**AGY06**], and Avila-Gouëzel [**AG13**], and from the analysis of $\mathrm{SL}(2,\mathbb{R})$ unitary representations (as in Flaminio-Forni [**FF03**]) a result on the existence and smoothness of a unique zero-average solution of a Poisson equation, which will be key to the proofs of our main theorems.

Let $\{X, Y, \Theta\}$ be the standard generators of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$, formed by the generators of the geodesic flow (the diagonal subgroup), the orthogonal geodesic flow and the subgroup $\mathrm{SO}(2,\mathbb{R})$ of rotations (the maximal compact subgroup).

We follow closely the notation in Avila-Gouëzel, [**AG13**, Section 3.4], and Flaminio-Forni [**FF03**]. In particular, following their notation, and for ξ varying in the space Ξ of all unitary irreducible representations of $\mathrm{SL}(2,\mathbb{R})$, let H_ξ be a family of irreducible unitary representations. For us, we will be concerned with the following decomposition

$$L^2(X,\nu) \simeq \int_\Xi H_\xi dm(\xi) \tag{A.0.1}$$

where m a measure on Ξ. It is a well-known result in representation theory that for every $\xi \in \Xi$ the spectrum of the Casimir operator $\square := -(X^2 + Y^2 - \Theta^2)$ on H_ξ is the set

$$\sigma(\square) := \{\frac{1 - s(\xi)^2}{4}\},$$

and there exists a set $\mathcal{N}(\xi) \subset \mathbb{Z}$ such that the spectrum of the operator Θ is the set

$$\sigma(\Theta) = \{ik | k \in \mathcal{N}(\xi)\}.$$

(The set $\mathcal{N}(\xi) = \mathbb{Z}$ when ξ belongs to the principal and complementary series, and $\mathcal{N}(\xi) = \{k \geq n | k \in \mathbb{Z}\}$ or $\mathcal{N}(\xi) = \{k \leq -n | k \in \mathbb{Z}\}$ when ξ belongs to the discrete series of parameter $s(\xi) = \pm 2n - 1$ for $n \in \mathbb{N} \setminus \{0\}$).

Finally, taking into account the direct integral in A.0.1, we also have that the L^2 norm of a function $f : X \to \mathbb{R}$ is given as

$$\|f\|^2 = \int \|f_\xi\|^2_{H_\xi} dm(\xi). \tag{A.0.2}$$

Let, for $c > 0$,

$$\mathcal{L}_c = -(X^2 + Y^2 + c^2\Theta^2) - 2cY\Theta\,.$$

The operator $\mathcal{L}$ is the sum of a second order self-adjoint differential operator and a differential operator of order one. In fact, since $[\Theta, Y] = -X$ (hence in particular Θ and Y do not commute, so the computation of $(Y + c\Theta)^2$ has to be done with care), we also have

$$\mathcal{L}_c = -X^2 - (Y + c\Theta)^2 - cX \,. \tag{A.0.3}$$

Following the method of **[FF03]**, for each $\xi \in \Xi$, we consider a orthogonal basis $\{u_k\} \subset H_\xi$ for the generator Θ of the maximal compact subgroup:

$$\Theta u_k = iku_k \,, \quad \text{for all } k \in \mathcal{N}(\xi)\,.$$

The basis $\{u_k\}$ is orthonormal only in representations of the principal series $s(\xi) \in i\mathbb{R}$, but not in representations of the complementary and discrete series ($s(\xi) \in \mathbb{R}$). In fact, the norms of the basis vectors satisfy the recursive equation (see **[FF03]**, formula (26) and following formulas)

$$\|u_k\|^2 = \frac{2|k| - 1 - s(\xi)}{2|k| - 1 + \overline{s(\xi)}} \|u_{k-1}\|^2 \,, \tag{A.0.4}$$

for $k \in \mathcal{N}(\xi) \setminus \{0\}$ for representations ξ of the principal and complementary series, and for $k \in \mathcal{N}(\xi) \setminus \{n\}$ or $\mathcal{N}(\xi) \setminus \{-n\}$, that is, for $k > n$ or $k < -n$, for representations ξ of the discrete series of parameter $s(\xi) = 2n-1$. (Note that in **[FF03]** the representation parameter $s(\xi)$ is denoted ν).

We then compute formulas for the vector field Y in the basis $\{u_k\}$. These formulas are not explicitly given in **[FF03]**, but can be easily derived from those for X, since $Y = [\Theta, X]$. We have (see **[FF03]**, Lemma 3.4)

$$Xu_k = \frac{2k + 1 + s(\xi)}{4} u_{k+1} - \frac{2k - 1 - s(\xi)}{4} u_{k-1} \,, \quad \text{for } k \in \mathcal{N}(\xi)$$

(for ξ a representation of the discrete series the above formula should be read with the convention that $u_k = 0$ for $k \notin \mathcal{N}(\xi)$).

From the commutation relation $[\Theta, X] = Y$ we derive the formula:

$$Yu_k = i\frac{2k + 1 + s(\xi)}{4} u_{k+1} + i\frac{2k - 1 - s(\xi)}{4} u_{k-1} \,, \quad \text{for } k \in \mathcal{N}(\xi)$$

Lemma A.1. *For every $c_0 > 1$ there exists a constant $K > 0$ such that for all $c \in [1, c_0]$ we have the lower bound*

$$\mathrm{Re}\langle \mathcal{L}_c f, f\rangle \geq K(\|Xf\|^2 + \|Yf\|^2 + \|\Theta f\|^2) \,, \quad \textit{for all } f \in \mathrm{dom}(\mathcal{L}_c)\,.$$

Proof. From the above formulas it follows that

$$\begin{aligned}
\langle Yf, \Theta f\rangle &= \sum_{k\in\mathcal{N}(\xi)} \frac{k(2k-1+s(\xi))}{4} f_{k-1}\overline{f}_k \|u_k\|^2 \\
&\qquad + \sum_{k\in\mathcal{N}(\xi)} \frac{k(2k+1-s(\xi))}{4} f_{k+1}\overline{f}_k \|u_k\|^2 \\
&= \sum_{k\in\mathcal{N}(\xi)} \frac{(k+1)(2k+1+s(\xi))}{4} f_k\overline{f}_{k+1} \|u_{k+1}\|^2 \\
&\qquad + \sum_{k\in\mathcal{N}(\xi)} \frac{k(2k+1-s(\xi))}{4} f_{k+1}\overline{f}_k \|u_k\|^2
\end{aligned}$$

We can then estimate as follows: for every $r > 0$, by the elementary inequality $(a-b)^2 \geq 0 \Leftrightarrow ab \leq (1/2)(a^2+b^2)$ for any $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
&\frac{|k+1||2k+1+s(\xi)|}{4} |f_k\overline{f}_{k+1}|\, \|u_{k+1}\|^2 \\
&\quad = \frac{|k+1||\overline{f}_{k+1}|\, \|u_{k+1}\|}{2} \frac{|2k+1+s(\xi)||f_k|\, \|u_{k+1}\|}{2} \\
&\quad \leq \frac{r(k+1)^2|f_{k+1}|^2\|u_{k+1}\|^2}{8} + \frac{r^{-1}|2k+1+s(\xi)|^2|f_k|^2\|u_{k+1}\|^2}{8}\,.
\end{aligned}$$

A similar estimate holds for the second term:

$$\begin{aligned}
&\frac{|k||2k+1-s(\xi)|}{4} |f_{k+1}\overline{f}_k|\, \|u_k\|^2 \\
&\qquad \leq \frac{rk^2|f_k|^2\|u_k\|^2}{8} + \frac{r^{-1}|2k+1-s(\xi)|^2|f_{k+1}|^2\|u_k\|^2}{8}
\end{aligned}$$

For representations of the principal series ($s(\xi) \in i\mathbb{R}$, and $\|u_k\| = 1$ for all $k \in \mathcal{N}(\xi) = \mathbb{Z}$) we immediately derive the bound

$$\begin{aligned}
|\langle Yf, \Theta f\rangle| &\leq \sum_{k\in\mathcal{N}(\xi)} \left(\frac{rk^2|f_k|^2}{8} + \frac{r(k+1)^2|f_{k+1}|^2}{8}\right) \\
&\quad + \sum_{k\in\mathcal{N}(\xi)} \frac{r^{-1}|2k+1+s(\xi)|^2|f_k|^2}{8} \\
&\quad + \sum_{k\in\mathcal{N}(\xi)} \frac{r^{-1}|2k+1-s(\xi)|^2|f_{k+1}|^2}{8}\,,
\end{aligned}$$

hence

$$\begin{aligned}
|\langle Yf, \Theta f\rangle| &\leq \sum_{k\in\mathcal{N}(\xi)} \frac{rk^2|f_k|^2}{4} + \sum_{k\in\mathcal{N}(\xi)} \frac{r^{-1}(4k^2+1+|s(\xi)|^2)|f_k|^2}{4} \\
&= \left(\frac{r}{4} + \frac{1}{r}\right) \|\Theta f\|^2 + \left(\frac{1-s(\xi)^2}{4r}\right) \|f\|^2\,.
\end{aligned}$$

In conclusion, for the principal series we have the estimates

$$\operatorname{Re}\langle \mathcal{L}_c f, f\rangle \geq \left(2 + c^2 - 2c\left(\frac{r}{4}+\frac{1}{r}\right)\right)\|\Theta f\|^2 + \left(1-\frac{2c}{r}\right)\left(\frac{1-s(\xi)^2}{4}\right)\|f\|^2\,,$$

which for $r = 2\rho c$ $(\rho > 1)$ gives

$$\operatorname{Re}\langle \mathcal{L}_c f, f\rangle \geq (\rho-1)\left(\frac{1}{\rho-1}+\frac{1}{\rho}-c^2\right)\|\Theta f\|^2 + \left(1-\frac{1}{\rho}\right)\left(\frac{1-s(\xi)^2}{4}\right)\|f\|^2\,.$$

Since by definition the Casimir operator $\Box = -X^2 - Y^2 + \Theta^2$, we have that

$$\left(\frac{1-s(\xi)^2}{4}\right)\|f\|^2 = \langle \Box f, f\rangle = \|Xf\|^2 + \|Yf\|^2 - \|\Theta f\|^2\,,$$

hence the statement is proved for representations of the principal series as a consequence of the above bound, by taking $\rho > 1$ such that $1/(\rho-1) > c^2$.

For representations of the complementary and discrete series (when $s(\xi) \in \mathbb{R}$), by formula (A.0.4) it follows that

$$(2k+1+s(\xi))^2\|u_{k+1}\|^2 = [(2k+1)^2 - s(\xi)^2]\|u_k\|^2\,;$$
$$(2k+1-s(\xi))^2\|u_k\|^2 = [(2k+1)^2 - s(\xi)^2]\|u_{k+1}\|^2\,.$$

From the above formulas we conclude

$$\begin{aligned}|\langle Yf, \Theta f\rangle| \leq &\sum_{k\in\mathcal{N}(\xi)} \frac{rk^2|f_k|^2\|u_k\|^2}{8} + \frac{r(k+1)^2|f_{k+1}|^2\|u_{k+1}\|^2}{8} \\ &+ \sum_{k\in\mathcal{N}(\xi)} \frac{r^{-1}[(2k+1)^2 - s(\xi)^2]|f_k|^2\|u_k\|^2}{8} \\ &+ \sum_{k\in\mathcal{N}(\xi)} \frac{r^{-1}[(2k+1)^2 - s(\xi)^2]|f_{k+1}|^2\|u_{k+1}\|^2}{8}\,,\end{aligned}$$

hence finally

$$\begin{aligned}|\langle Yf, \Theta f\rangle| &\leq \sum_{k\in\mathcal{N}(\xi)} \frac{rk^2|f_k|^2\|u_k\|^2}{4} + \sum_{k\in\mathcal{N}(\xi)} r^{-1}\left(k^2 + \frac{1-s(\xi)^2}{4}\right)|f_k|^2\|u_k\|^2 \\ &= \left(\frac{r}{4}+\frac{1}{r}\right)\|\Theta f\|^2 + \left(\frac{1-s(\xi)^2}{4r}\right)\|f\|^2\,.\end{aligned}$$

It follows that

$$\operatorname{Re}\langle \mathcal{L}_c f, f\rangle \geq \left(2 + c^2 - 2c\left(\frac{r}{4}+\frac{1}{r}\right)\right)\|\Theta f\|^2 + \left(\frac{1-s(\xi)^2}{4}\right)\left(1-\frac{2c}{r}\right)$$

For $1 - s(\xi)^2 > 0$ (complementary series), we can take $r = 2\rho c$ to get

$$\operatorname{Re}\langle \mathcal{L}_c f, f\rangle \geq (\rho-1)\left(\frac{1}{\rho-1}+\frac{1}{\rho}-c^2\right)\|\Theta f\|^2 + \left(1-\frac{1}{\rho}\right)\left(\frac{1-s(\xi)^2}{4}\right)\|f\|^2\,,$$

and the statement follows exactly as in the case of the principal series.

For $1 - s(\xi)^2 < 0$ (discrete series) we can take $r = 1$ to get, for $c \geq 1$,

$$\mathrm{Re}\langle \mathcal{L}_c f, f\rangle \geq \frac{7}{16}\|\Theta f\|^2 + \left(\frac{s(\xi)^2 - 1}{4}\right)\|f\|^2 .$$

For irreducible representations of the discrete series the Casimir operator has negative value, hence the statement follows as a consequence of the bound:

$$\|\Theta f\|^2 > \|Xf\|^2 + \|Yf\|^2 .$$

Since $1/(\rho - 1)$ diverges as $\rho \to 1^+$, for every $c_0 > 1$ there exists $\rho_0 > 1$ such that $1/(\rho_0 - 1) > 1 + c_0^2$, hence the argument is complete. □

Let $W^{2,2}(X,\nu)$ denote the (foliated) Sobolev space on X, that is, the space of functions $f \in L^2(X,\nu)$ such that for all $\mathcal{V}, \mathcal{W} \in \mathfrak{sl}(2,\mathbb{R})$

$$\mathcal{V} f \quad \text{and} \quad \mathcal{V}\mathcal{W} f \in L^2(X,\nu) .$$

Corollary A.2. *The operators $\mathcal{L}_c$ are closed on the domain $W^{2,2}(X,\nu)$ and have bounded inverse on the orthogonal complement of the space of constant functions. For $F \in L^2(X,\nu)$ with $\int_X F d\mu = 0$, we can find a unique zero average solution $U \in W^{2,2}(X,\nu)$ of the equation $\mathcal{L}_c U = F$.*

Proof. By [**AGY06**], [**AG13**] the $\mathrm{SL}(2,\mathbb{R})$ Laplacian $\Delta = -(X^2 + Y^2)$ has a spectral gap, hence there exists $\mu_0(X) > 0$ such that, by the theory of unitary representations, for spherical representations, that is for representations $\xi \in \Xi$ of the principal ($s(\xi) \in \imath\mathbb{R}$) and complementary ($s(\xi) \in (-1,1)$) series, we have the bound

$$\frac{1 - s(\xi)^2}{4} \geq \mu_0(X) > 0 ,$$

hence by Lemma A.1 the operator $\mathcal{L}_c$ is bounded below on all non-trivial spherical representations:

$$\langle \mathcal{L}_c f, f\rangle \geq K\left(\langle \Box f, f\rangle + 2\|\Theta f\|^2\right) \geq K\left(\mu_0(X)\|f\|^2 + 2\|\Theta f\|^2\right) .$$

For representations of the discrete series we have

$$\langle \mathcal{L}_c f, f\rangle \geq K\left(\langle \Box f, f\rangle + 2\|\Theta f\|^2\right) \geq K\|\Theta f\|^2 \geq \frac{K}{2}\left(\|f\|^2 + \|\Theta f\|^2\right) .$$

From the above bounds it follows that $\mathcal{L}_c$ has bounded inverse on the orthogonal complement of the trivial representation. From the above bounds it follows immediately that for all $f \in \mathrm{dom}(\mathcal{L}_c)$

$$\mathcal{V} u \in L^2(X,\nu) , \qquad \text{for all } \mathcal{V} \in \mathfrak{sl}(2,\mathbb{R}) .$$

Since $\mathcal{L}_c = -X^2 - (Y + c\Theta)^2 - X$, it follows that the domain $\mathrm{dom}(\mathcal{L}_c)$ of the operator $\mathcal{L}_c$ coincides with the domain of the formal adjoint $\mathcal{L}_c^{\#} = -X^2 - (Y + c\Theta)^2 + X$. The symmetric operator

$$A_c = \frac{\mathcal{L}_c + \mathcal{L}_c^{\#}}{2} = -X^2 - (Y + c\Theta)^2$$

is positive and by Lemma A.1 satisfies

$$\langle A_c f, f\rangle \geq K\left(\|Xf\|^2 + \|Yf\|^2 + \|\Theta f\|^2\right) .$$

Let $\hat\Delta := -(X^2 + Y^2 + \Theta^2)$ denote the $\mathrm{SL}(2,\mathbb{R})$ Laplacian. The space $W^{2,2}(X,\nu)$ is the maximal domain of the operator $\hat\Delta$ and it is a Hilbert space with its graph norm. The hermitian form defined by the operator A_c (on the orthogonal complement of the trivial representation) is well-defined, bounded and coercive on $W^{2,2}(X,\nu)$, hence by the representation theorem there exists a bounded invertible operator T on $W^{2,2}(X,\nu)$ such that

$$\langle A_c f, g\rangle = \langle \hat\Delta T f, g\rangle\,, \qquad \text{for all } f,g \in W^{2,2}(X,\nu)\,.$$

It follows that the operator A_c is closed on the domain $W^{2,2}(X,\nu)$, hence it is self-adjoint with domain $W^{2,2}(X,\nu)$ as $\mathrm{Ker}(A_c \pm iI) = \{0\}$ by Lemma A.1.

In fact, if $f_n \to f$ in $W^{2,2}(X,\nu)$ and $A_c f_n \to g$ in $L^2(X,\nu)$, it follows that $Tf_n \to Tf$ in $W^{2,2}(X,\nu)$ and $\hat\Delta(Tf_n) = A_c(f_n) \to\to g$ in $L^2(X,\nu)$. Since $\hat\Delta$ is closed on $W^{2,2}(X,\nu)$, it follows that $Tf \in W^{2,2}(X,\nu)$ and $\hat\Delta(Tf) = g$, and finally $f = T^{-1}(Tf) \in W^{2,2}(X,\nu)$ and $A_c(f) = \hat\Delta(Tf) = g$.

Since A_c is closed and self-adjoint on $W^{2,2}(X,\nu)$, it follows that the maximal domain of A_c, which coincides with the maximal domain of $\mathcal{L}_c$ and $\mathcal{L}_c^{\#} = \mathcal{L}_c^*$ is equal to the space $W^{2,2}(X,\nu)$.

□

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